MATH 525a SAMPLE MIDTERM SOLUTIONS
FALL 2016
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(1) For $x < y$ in A, we have $s(x) \leq y$ (by definition of successor), so I_x and I_y are disjoint. Hence $\{I_x : x \in A\}$ is a collection of disjoint open intervals so is at most countable (because each interval contains a rational.)

(2)(a) A is μ^*-measurable if for every $E \subset X$, we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
(b) In the text.

(3) Let $\epsilon > 0$. By uniform integrability, there exists $\delta > 0$ such that

$$\mu(A) < \delta \implies \int_A |f_n| \, d\mu < \epsilon \quad \text{for all } n. \tag{1}$$

Let $E_n = \{x : |f_n(x)| > \epsilon\}$. Since $f_n \to 0$ in measure, we have $\mu(E_n) \to 0$. Therefore for large n we have $\mu(E_n) < \delta$ and hence by (1), $\int_{E_n} |f_n| < \epsilon$. Therefore

$$\begin{aligned}\int |f_n| &= \int_{E_n} |f_n| + \int_{E_n^c} |f_n| \\ &< \epsilon + \int_{E_n^c} \epsilon \, d\mu \\ &= \epsilon + \epsilon\mu(E_n^c) \\ &\leq \epsilon + \epsilon\mu(X).\end{aligned}$$

Since ϵ is arbitrary, this shows $\int |f_n| \to 0$.

(4) Suppose $E_1, E_2, \dots \in \mathcal{C}$ and let $\epsilon > 0$. For each j there exists $A_j \in \mathcal{A}$ with $A_j \supset E_j$ and $\mu(A_j \backslash E_j) < \epsilon/2^j$. For a given N we can approximate $\cap_{j=1}^\infty E_j$ by $\cap_{j=1}^N A_j$ as follows: we have

$$\cap_{j=1}^\infty E_j \subset \cap_{j=1}^N E_j \subset \cap_{j=1}^N A_j,$$

so

$$\begin{aligned}\mu\Big((\cap_{j=1}^N A_j)\backslash(\cap_{j=1}^\infty E_j)\Big) &= \mu\Big((\cap_{j=1}^N A_j)\backslash(\cap_{j=1}^N E_j)\Big) + \mu\Big((\cap_{j=1}^N E_j)\backslash(\cap_{j=1}^\infty E_j)\Big) \\ &\leq \sum_{j=1}^N \mu(A_j \backslash E_j) + \mu\Big((\cap_{j=1}^N E_j)\backslash(\cap_{j=1}^\infty E_j)\Big) \\ &< \sum_{j=1}^N \frac{\epsilon}{2^j} + \mu\Big((\cap_{j=1}^N E_j)\backslash(\cap_{j=1}^\infty E_j)\Big) \qquad (2) \\ &< \epsilon + \mu\Big((\cap_{j=1}^N E_j)\backslash(\cap_{j=1}^\infty E_j)\Big).\end{aligned}$$

The sets $(\cap_{j=1}^{N} E_j)\backslash(\cap_{j=1}^{\infty} E_j)$ decrease to ϕ as $N \to \infty$, and $\mu(X) < \infty$, so by continuity from above,

$$\mu\bigg((\cap_{j=1}^{N} E_j)\backslash(\cap_{j=1}^{\infty} E_j)\bigg) \to 0.$$

Therefore for N large we have $\mu\bigg((\cap_{j=1}^{N} E_j)\backslash(\cap_{j=1}^{\infty} E_j)\bigg) < \epsilon$. Putting this in (2) we get

$$\mu\bigg((\cap_{j=1}^{N} A_j)\backslash(\cap_{j=1}^{\infty} E_j)\bigg) < 2\epsilon.$$

Since ϵ is arbitrary and $\cap_{j=1}^{N} A_j \in \mathcal{A}$, this shows $\cap_{j=1}^{\infty} E_j$ is approximable from outside by $\mathcal{A}$.