MATH 525a SAMPLE MIDTERM EXAM
Fall 2016
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(1) Recall that $\mathbb{R}$ is not well-ordered under its standard ordering "$\le$", but some of its subsets are well-ordered by this ordering, for example the positive integers.

Suppose $A \subset \mathbb{R}$ and A is well-ordered under the standard ordering "$\le$" of real numbers. Show that A is at most countable.

HINT: The *successor* of $x \in A$, call it $s(x)$, is the least element of $\{y \in A : y > x\}$. (Here we let $s(x) = \infty$ if this set is empty.) For $x \in A$ there is an interval $I_x = (x, s(x))$. How are these intervals related to each other?

(2)(a) For μ^* an outer measure on subsets of X, state what it means for a set $A \subset X$ to be μ^*-*measurable*. (That is, give the definition.)

(b) Prove Theorem 1.13(b): If μ_0 is a premeasure on an algebra $\mathcal{A}$ and

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \cup_{j=1}^{\infty} A_j \right\},$$

then every set in $\mathcal{A}$ is μ^*-measurable. [Note this theorem was on the list to know, when this exam was given.]

(3) A sequence $\{f_n\}$ of functions is called *uniformly integrable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta \implies \int_A |f_n| \, d\mu < \epsilon$ for all n. (That is, the same δ works for all n.)

Suppose $\mu(X) < \infty$, $\{f_n\}$ is uniformly integrable and $f_n \to 0$ in measure. Show that $\int |f_n| \to 0$.

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $\mathcal{A} \subset \mathcal{M}$ be an algebra (not a σ-algebra!) A set $E \in \mathcal{M}$ is called *approximable from outside by* $\mathcal{A}$ if for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ with $A \supset E, \mu(A \backslash E) < \epsilon$.

Show that $\mathcal{C} = \{E \in \mathcal{M} : E \text{ is approximable from outside by } \mathcal{A}\}$ is closed under countable intersections.

Exam Information

On the actual exam, one problem (or part of a problem) will be to prove one of the following theorems: 1.10 (countable subadditivity part), 1.18 (the sup part), 2.6, 2.16, 2.18, 2.24, 2.29. You do not have to give the proof as done in the text–you just need a valid proof. If the text proof refers to some numbered Lemma, Theorem, etc., you don't have to know the number, you can just write, "By an earlier result,..." or something similar.

The exam will cover up through Section 2.5.

The emphasis is mostly on problems with SHORT solutions, which means proofs that are closely based on using the definitions and/or theorem statements, not a lot of multi-step logic. So be sure you know the definitions and theorem statements thoroughly!

It is not expected that you can necessarily do all the problems in the 50-55 minutes available, if they are like the four problems above. It varies from one exam to another, but in the case of the sample exam above, 2-3 problems done mostly correctly would be a good performance.