MATH 525a ASSIGNMENT 9
FALL 2016
Prof. Alexander
Due Wednesday November 30.

Chapter 3, p. 107 #33, 37
Chapter 5, p. 154 #1, 4.
Also:

(I) Let $F : [0,1] \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \leq |x^{1/2} - y^{1/2}|$$

for all $x, y \in [0,1]$. Is F necessarily absolutely continuous? Justify your answer.

(II) It can be shown, and you may assume, that there exists a measurable set $E \subset [0,1]$ with the following property: for every interval $I \subset [0,1]$ with $m(I) > 0$, we have $0 < m(E \cap I) < m(I)$. Let $F(x) = m(E \cap [0,x])$ for $x \in [0,1]$. Show that F is absolutely continuous and strictly increasing, but F^{-1} is not absolutely continuous.

(III) Show that there exists an increasing function F on $\mathbb{R}$ such that $F' = 0$ a.e. but F is not constant on any open interval.

(IV) Suppose f is absolutely continuous on $[0,1]$, and $A \subset [0,1]$ has Lebesgue measure 0. Show that $f(A)$ has Lebesgue measure 0.

(V) If f and g are absolutely continuous on $[a,b]$, show that fg is absolutely continuous.

HINTS:

(33) Note F is not asumed right-continuous, so there might not be a corresponding μ_F. Also, it's enough to prove the inequality with $b - \epsilon$ in place of b on the right side, for all ϵ.

(37) To show the existence of M implies absolute continuity, use the definition of absolute continuity.

(I) This should be quite short.

(II) This one is somewhat difficult, but give it a reasonable try. You need to use the ball-covering lemma. Choose the balls, centered at each point of $[0,1]\backslash E$, using the fact that when $F'(x) = 0$, there is a neighborhood of x such that $|F(y) - F(x)|$ is small relative to $|y - x|$ for all y in the neighborhood. This can be reformulated as a statement about increments of F^{-1}.

(III) What properties must μ_F have? Find some μ_F with these properties.

(IV) For (a, b) an interval, let m and M be the inf and sup, respectively, of f on $[a, b]$, and let $c \leq d$ be points in $[a, b]$ where this inf and sup are achieved. (Either sup or inf could be at c, it doesn't matter.) Then $m(f((a, b))) = M - m = |f(d) - f(c)|$.