MATH 525a ASSIGNMENT 8
FALL 2016
Prof. Alexander
Due Wednesday November 16.

Chapter 3 #11, 12, 16, 19(second part only—show $\nu \ll \lambda \iff |\nu| \ll \lambda$), 25 and:

(I)(a) Let m be Lebesgue measure. Suppose ν is a finite measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and $\alpha = \sup\left\{\frac{\nu(E)}{m(E)} : E \in \mathcal{M}, m(E) > 0\right\} < \infty$. Show that ν is absolutely continuous with respect to m.

(b) For $A, B \in \mathcal{B}_{\mathbb{R}}$ we say A is an *m-essential subset* of B if there exists $N \in \mathcal{B}_{\mathbb{R}}$ with $m(N) = 0$ and $A \backslash N \subset B$. Suppose there exists at least one set which achieves the supremum in (b). Show that the sets on which the supremum is achieved can be characterized as follows: there exists a measurable set Y such that the supremum is achieved on F if and only if F is an m-essential subset of Y. (In other words, up to null sets, Y is the largest set on which the sup is achieved.)

(c) Give an example of a ν for which $d\nu/dm$ is bounded but the supremum in (a) is not achieved.

(II) Let us say that two signed measures ν_1, ν_2 on $(X, \mathcal{M})$ are *compatible* if there exists a decomposition $X = P \cup N$ which is a Hahn decomposition for both ν_1 and ν_2. According to Proposition 3.14 page 94, for ν_1, ν_2 finite signed measures,

$$(*) \qquad |\nu_1 + \nu_2|(E) \leq |\nu_1|(E) + |\nu_2|(E) \quad \text{for all } E.$$

Let $\mu = |\nu_1| + |\nu_2|$ and let $f_j = \frac{d\nu_j}{d\mu}, j = 1, 2$. We write $\{f_j > 0\}$ as a shorthand for $\{x \in X : f_j(x) > 0\}$, and similarly for $\{f_j < 0\}$. Show that the following are equivalent:

(i) ν_1 and ν_2 are compatible;

(ii) equality holds in (*) for all E;

(iii) $\mu(\{f_1 > 0\} \cap \{f_2 < 0\}) = \mu(\{f_1 < 0\} \cap \{f_2 > 0\}) = 0$.

(III) Let μ be a finite signed measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with $\mu \ll m$, let E be a Borel set in $\mathbb{R}$ and let $G(x) = \mu((-\infty, x] \cap E)$. Here m denotes Lebesgue measure. Show that a.e. in E^c, $G' = 0$. (This includes showing that G' exists a.e. in E^c.)

(IV) A measure ρ is called *semifinite* if for every measurable set E with $\rho(E) = \infty$, there is a measurable $F \subset E$ with $0 < \rho(F) < \infty$. Suppose $0 < f < \infty$ and $d\nu = f\, d\mu$. Show that if ν is semifinite, then μ is semifinite.

HINTS:

(11)(a) Why is a *single* function uniformly integrable?

(b) $|f_n| \leq |f| + |f_n - f|$.

(12) In general, to show $d\alpha/d\beta = f$ for some particular f, you must show $\alpha(E) = \int_E f\, d\beta$ for all E. It may be enough to verify this just for a limited class of E's.

(16) What is $\frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda}$? Is $1 - f$ a Radon-Nikodym derivative of something? Show

$$f = \frac{d\nu}{d\mu}(1 - f).$$

(25)(a) This is very quick and easy if you look at it correctly.

(b) Consider of corner of a set, say in the plane. An example where $D_E(x)$ doesn't exist is more difficult (I think)—construct a set $E \subset \mathbb{R}$ such that the ratio in the definition of $D_E(0)$ oscillates between two values as $r \to 0$.

(I)(b) This is a tricky one—try it but don't spend forever on it! What can you say about the values of the function $d\nu/dm$ on the set Y? So, describe Y in terms of $d\nu/dm$ values. What happens if $d\nu/dm$ is unbounded?

(II) Show (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii). To show (ii) implies (iii), show "not (iii)" implies "not (ii)."

(III) This is a short application of one of the main theorems.