MATH 525a ASSIGNMENT 7 SOLUTIONS
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Chapter 2

(46) For fixed y, $\chi_D(x,y) = 1$ only for one x, so $\int \chi_D(x,y)\ \mu(dx) = 0$. Hence $\int\int \chi_D\ d\mu\ d\nu = 0$.

For fixed x, $\chi_D(x,y) = 1$ for exactly one y, so $\int \chi_D(x,y)\ \nu(dy) = 1$. Hence $\int\int \chi_D\ d\nu\ d\mu = 1$.

Now

$$(*)\qquad (\mu\times\nu)(D) = (\mu\times\nu)^*(D) = \inf\left\{\sum_{j=1}^\infty (\mu\times\nu)(E_j) : D\subset \cup_j E_j, E_j\in\mathcal{A} \text{ for all } \mathcal{A}\right\},$$

where $\mathcal{A}$ is the algebra {all finite unions of disjoint rectangles}. In fact the inf doesn't change if we restrict each E_j to be just one rectangle. But for every rectangle $E_j = A_j\times B_j$ with card$(E_j\cap D) > 1$, A_j and B_j must be intervals of positive length so $(\mu\times\nu)(E_j) = \mu(A_j)\nu(B_j) = \mu(A_j)\cdot\infty = \infty$. Since D is uncountable, a least one E_j must have card$(E_j\cap D) > 1$, so $(\mu\times\nu)(E_j) = \infty$. Thus $\int \chi_D\ d(\mu\times\nu) = (\mu\times\nu)(D) = \infty$, by (*).

(48) Since $|f|\geq 0$, by Tonelli's Theorem (2.37a) we have

$$\int |f|\ d(\mu\times\nu) = \int\int f(m,n)\ \mu(dm)\ \nu(dn) = \int 2\ \nu(dn) = 2\nu(\mathbb{N}) = \infty.$$

But $\int f(m,n)\ \mu(dm) = 1 + (-1) = 0$ for all n, so $\int\int f\ d\mu\ d\nu = 0$, while

$$\int f(m,n)\ \nu(dn) = \begin{cases} 1 & \text{if } m = 1,\\ 1+(-1) = 0 & \text{if } m\geq 2\end{cases} = \chi_{\{1\}}(m),$$

so $\int\int f\ d\nu\ d\mu = \int \chi_{\{1\}}\ d\mu = 1$.

(51)(a) Let $h_1(x,y) = f(x), h_2(x,y) = g(y)$. For $E\in\mathcal{B}_\mathbb{C}$ we have $h_1^{-1}(E) = f^{-1}(E)\times\mathbb{C}\in\mathcal{M}\times\mathcal{N}, h_2^{-1}(E) = \mathbb{C}\times g^{-1}(E)\in\mathcal{M}\times\mathcal{N}$. Thus h_1, h_2 are $\mathcal{M}\times\mathcal{N}$-measurable. Hence by Proposition 2.6, so is $h = h_1h_2$.

(b) Let $\tilde{X} = \{x\in X : f(x)\neq 0\}, \tilde{Y} = \{y\in Y : g(y)\neq 0\}$. The the restrictions to $\tilde{X}$ and $\tilde{Y}$ are σ-finite, since $f,g\in L^1$. It is enough to consider f,g as functions on $\tilde{X},\tilde{Y}$, so we may assume X, Y are σ-finite. Then by Tonelli's Theorem (2.37a),

$$\begin{aligned}\int |h|\ d\nu &= \int\int |f(x)||g(y)|\ \mu(dx)\nu(dy)\\ &= \int |g(y)| \left(\int |f|\ d\mu\right)\ \nu(dy)\\ &= \left(\int |f|\ d\mu\right)\left(\int |g|\ d\nu\right)\\ &< \infty, \end{aligned} \tag{1}$$

so $h \in L^1(\mu \times \nu)$. Therefore we can repeat (**??**) without absolute values, which shows

$$\int h\ d(\mu \times \nu) = \left(\int f\ d\mu\right)\left(\int g\ d\nu\right).$$

Chapter 3

(2) Let $X = P \cup N$ be the Hahn decomposition. Then

$$\begin{aligned} E \text{ is } \nu\text{-null} &\iff \nu(F) = 0 \quad \text{for all (measurable) } F \subset E\\ &\iff \nu(F \cap P) = \nu(F \cap N) = 0 \quad \text{for all } F \subset E\\ &\iff \nu^+(F) = \nu^-(F) = 0 \quad \text{for all } F \subset E\\ &\iff \nu^+(E) = \nu^-(E) = 0\\ &\iff |\nu|(E) = 0. \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} \nu \perp \mu &\iff \text{there exists a } \mu\text{-null } F \text{ with } F^c\ \nu\text{-null (i.e. } \nu \text{ supported on } F)\\ &\iff (*) \text{ there exists a } \mu\text{-null } F \text{ with } F^c\ |\nu|\text{-null} \quad (|\nu|\text{-null same as } \nu\text{-null, by (??)})\\ &\iff |\nu| \perp \mu, \end{aligned}$$

while

$$\begin{aligned} (*) &\iff \text{there exists a } \mu\text{-null } F \text{ with } F^c\ \nu^+\text{-null and } \nu^-\text{-null}\\ &\iff \text{there exist } \mu\text{-null } G, H \text{ with } G^c\ \nu^+\text{-null and } H^c\ \nu^-\text{-null (take } F = G \cup H)\\ &\iff \nu^+ \perp \mu \text{ and } \nu^- \perp \mu. \end{aligned}$$

Thus $\nu \perp \mu \iff \nu^+ \perp \mu$ and $\nu^- \perp \mu$.

(4) Suppose $\nu = \lambda - \mu$ with λ, μ positive. Let $X = P \cup N$ be the Hahn decomposition of ν. For $E \subset P$ we have

$$\nu^+(E) = \nu(E) = \lambda(E) - \mu(E) \leq \lambda(E),$$

so for general $F \subset X$, taking $E = F \cap P$,

$$\nu^+(F) = \nu^+(F \cap P) \leq \lambda(F \cap P) \leq \lambda(F).$$

Similarly for $E \subset N$,

$$\nu^-(E) = -\nu(E) = \mu(E) - \lambda(E) \leq \mu(E)$$

so for general $F \subset X$, taking $E = F \cap N$,

$$\nu^-(F) = \nu^-(F \cap N) \leq \mu(F \cap N) \leq \mu(F).$$

Thus $\nu^+ \leq \lambda, \nu^- \leq \mu$.

(A) Let $0 < M < \infty$. By Fubini-Tonelli we have

$$\int_{[-M,M]} f \ dm = \int_{\mathbb{R}} \int_{[-M,M]} \frac{1}{|x-y|^{1/2}} \ dm(x) \ d\mu(y).$$

Let

$$g(y) = \int_{-M}^{M} \frac{1}{|x-y|^{1/2}} \ dx.$$

For $y \in [-M, M]$ we have

$$\begin{aligned} g(y) &= \int_{-M}^{y} \frac{1}{(y-x)^{1/2}} \ dx + \int_{y}^{M} \frac{1}{(x-y)^{1/2}} \ dx \\ &= 2(y+M)^{1/2} + 2(M-y)^{1/2} \\ &\leq 4(2M)^{1/2}. \end{aligned}$$

For $y > M$ the integrand in the definition of g is decreasing in y. Hence g is decreasing, so $g(y) \leq g(M) \leq (2M)^{1/2}$. Similarly, for $y < -M$ we have $g(y) \leq g(-M) \leq (2M)^{1/2}$. Thus g is a bounded measurable function on $\mathbb{R}$, so it is μ-integrable. Therefore

$$\int_{[-M,M]} f \ dm = \int_{\mathbb{R}} g \ d\mu < \infty,$$

so f is finite m-a.e.

(B) Let $\tilde{X} = \{x \in X : f(x) > 0\}$ and let $\tilde{\mu}$ be the restriction of μ to measurable subsets of $\tilde{X}$. Then by the assumptions in the problem, $\tilde{\mu}$ is a σ-finite measure.. The results of the problem are unchanged if we replace X, μ with $\tilde{X}, \tilde{\mu}$ so we may assume μ is σ-finite.

Using Fubini-Tonelli we have

$$\begin{aligned}\int_X f\ d\mu &= \int_X \int_{(0,f(x))} m(dt)\ \mu(dx) \\ &= \int_X \int_{(0,\infty)} \chi_{\{0<t<f(x)\}}\ m(dt)\ \mu(dx) \\ &= \int_{(0,\infty)} \int_X \chi_{\{0<t<f(x)\}}\ \mu(dx)\ m(dt) \\ &= \int_{(0,\infty)} \mu(E_t)\ m(dt).\end{aligned}$$

Since $\mu(E_t) < \infty$ for all $t > 0$, we can apply continuity from above to conclude $g(t) \to 0$ as $t \to \infty$. Therefore using Fubini-Tonelli again,

$$\begin{aligned}\int_{(0,\infty)} t\ \lambda_g(dt) &= \int_{(0,\infty)} \int_{(0,t)} m(dx)\ \lambda_g(dt) \\ &= \int_{(0,\infty)} \int_{(0,\infty)} \chi_{\{x\leq t\}}\ m(dx)\ \lambda_g(dt) \\ &= \int_{(0,\infty)} \lambda_g((x,\infty))\ m(dx) \\ &= \int_{(0,\infty)} -g(x)\ m(dx) \\ &= \int_{(0,\infty)} \mu(E_x)\ m(dx).\end{aligned}$$

(C) Let $X = P \cup N$ be the Hanh decomposition of ν. We have

$$\int f\ d\nu = \int f^+\ d\nu^+ - \int f^-\ d\nu^+ - \int f^+\ d\nu^- + \int f^-\ d\nu^-.$$

These four integrals are all nonnegative, and by assumption at most one is infinite. Hence

$$\left|\int f\ d\nu\right| \leq \int f^+\ d\nu^+ + \int f^-\ d\nu^+ + \int f^+\ d\nu^- + \int f^-\ d\nu^- = \int |f|\ d\nu^+ + \int |f|\ d\nu^- = \int |f|\ d|\nu|.$$

(D) Let $X = P \cup N$ be the Hanh decomposition of ν. Suppose $|f| \leq 1$. Then by the above problem (C),

$$\left|\int_A f\ d\nu\right| = \left|\int f\chi_A\ d\nu\right| \leq \int |f|\chi_A\ d|\nu| \leq \int \chi_A\ d|\nu| = |\nu|(A).$$

Therefore $\sup\left\{\left|\int_A f\ d\nu\right| : |f| \leq 1\right\} \leq |\nu|(A)$. In the other direction, given A let $f = \chi_{A\cap P} - \chi_{A\cap N}$. Then $|f| \leq 1$ and

$$\int_A f\ d\nu = \int_{A\cap P} f\ d\nu + \int_{A\cap N} f\ d\nu = \nu(A\cap P) - \nu(A\cap N) = |\nu|(A).$$

Therefore $\sup\left\{\left|\int_A f\ d\nu\right| : |f| \leq 1\right\} \geq |\nu|(A)$, meaning we have equality.

(E) Applying Theorem 2.37(a) to $f = \chi_E$ we get

$$\int \nu(E_x)\ d\mu(x) = \int \mu(E^y)\ d\nu(y) = 0,$$

and therefore $\nu(E_x) = 0$ for μ-a.e. x.