MATH 525a ASSIGNMENT 7
FALL 2016
Prof. Alexander
Due Wednesday November 2.

Chapter 2 #46, 48, 51
Chapter 3 #2, 4 and

(A) Let μ be a finite Borel measure on $\mathbb{R}$ and let

$$f(x) = \int_{\mathbb{R}} \frac{d\mu(y)}{|x-y|^{1/2}}.$$

Here $\frac{1}{|x-y|^{1/2}}$ should be interpreted as $+\infty$ when $x = y$. Prove that f is finite a.e. with respect to Lebesgue measure on $\mathbb{R}$.

(B) Let f be a nonnegative measurable function on $(X, \mathcal{M}, \mu)$ with $\mu(\{x : f(x) > t\} < \infty$ for all $t > 0$. Let $E_t = \{x : f(x) > t\}$ and $g(t) = -\mu(E_t)$. Let λ_g be the Lebesgue-Stieltjes measure induced by g, that is, the measure satisfying

$$\lambda_g((a,b]) = g(b) - g(a) \quad \text{for all } 0 < a < b.$$

Let m denote Lebesgue measure. Show that

$$\int f \, d\mu = \int_{(0,\infty)} \mu(E_t)\, m(dt) = \int_{(0,\infty)} t\, \lambda_g(dt).$$

(C) For a signed measure ν, define $\int f\, d\nu = \int f\, d\nu^+ - \int f\, d\nu^-$, whenever the integrals on the right both exist and are not both infinite. Show that $|\int f\, d\nu| \le \int |f|\, d|\nu|$.

(D) For a signed measure ν, show that

$$|\nu|(A) = \sup\left\{ \left| \int_A f\, d\nu \right| : |f| \le 1 \right\}.$$

(E) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and $E \in \mathcal{M} \otimes \mathcal{N}$, with sections E_x and E^y. Suppose $\mu(E^y) = 0$ for ν-almost every y. Show that $\nu(E_x) = 0$ for μ-almost every x.

HINTS:

(46) To compute $(\mu \times \nu)(D)$, use the fact that $(\mu \times \nu)(D) = (\mu \times \nu)^*(D)$.

(51)(a) Use Prop. 2.6.
(b) One approach is to approximate by simple functions.

(4) To get $\lambda \geq \nu^+$, first prove it for subsets of P, where $X = P \cup N$ is the Hanh decomposition for ν.

(A) How can you turn this into an application of the Fubini-Tonelli Theorem? Also, consider $[-M, M]$ in place of $\mathbb{R}$.

(B) For the first equality, use $a = \int_{(0,a)} m(dx)$ for appropriate $a \geq 0$, and then use Fubini-Tonelli. For the second equality, use this same idea differently.