MATH 525a ASSIGNMENT 6 SOLUTIONS
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Chapter 2

(33) By definition of lim inf, there is a subsequence $\{f_{n_k}\}$ for which $\int f_{n_k} \to \liminf \int f_n$. Since $f_{n_k} \to f$ in measure, there exists a further subsequence $\{f_{m_j}\}$ of $\{f_{n_k}\}$ for which $f_{m_j} \to f$ a.e. By Fatou's Lemma,

$$\int f = \int \liminf_j f_{m_j} \leq \liminf_j \int f_{m_j} = \liminf_n \int f_n.$$

(34)(a) We claim that every subsequence of $\{\int f_n\}$ has a further subsequence converging to $\int f$. (It is a basic real-analysis fact about sequences of real numbers that this implies that the full sequence $\int f_n \to \int f$.) To prove the claim, let $\{n_k\}$ be a subsequence. Since $f_{n_k} \to f$ in measure, by Theorem 2.30 there exists a further subsequence $\{f_{m_j}\}$ of $\{f_{n_k}\}$ for which $f_{m_j} \to f$ a.e. By Dominated Convergence, $\int f_{m_j} \to \int f$, proving the claim.

(b) There exists a subsequence $f_{n_k} \to f$ a.e. Since $|f_{n_k}| \leq g$, it follows that $|f| \leq g$. Therefore $|f_n - f| \leq |f_n| + |f| \leq 2g$.

Now let $\{f_{n_k}\}$ be any subsequence. There exists a further subsequence $f_{m_j} \to f$ a.e., that is, $|f_{m_j} - f| \to 0$ a.e. By Dominated Convergence, $\int |f_{m_j} - f| \to 0$. Thus every subsequence of $\{\int |f_n - f|\}$ has a further subsequence converging to 0. As in (a) this implies $\int |f_n - f| \to 0$.

(35) By definition of convergence in measure, if $f_n \to f$ in measure then for every $\epsilon > 0$ and $\delta > 0$ there exists N such that $n \geq N$ implies $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \delta$. In particular we can take $\delta = \epsilon$.

Conversely suppose the above is valid in the case $\epsilon = \delta$. Let $\epsilon, \delta > 0$ (not necessarily equal) and let $u = \min(\delta, \epsilon)$. Then there exists N such that $n \geq N$ implies

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| > u\} < u \leq \delta.$$

Here the first inequality follows from $\epsilon \geq u$. This shows that $f_n \to f$ in measure.

(37)(a) Let $N = \{x : f_n(x) \not\to f(x)\}$, so N is null. If $x \notin N$ then $\phi(f_n(x)) \to \phi(f(x))$. Thus $\phi \circ f_n \to \phi \circ f$ a.e.

(b) Suppose $f_n \to f$ uniformly. Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|w - z| < \delta \quad \text{implies} \quad |\phi(w) - \phi(z)| < \epsilon. \qquad (*)$$

Also, there exists N such that $n \geq N$ implies $|f_n(x) - f(x)| < \delta$ for all x. Therefore $n \geq N$ implies $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$ for all x, which means $\phi \circ f_n \to \phi \circ f$ uniformly.

Next suppose $f_n \to f$ almost uniformly. Let $\epsilon > 0$, and let $E \subset X$ with $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on E^c. By the above, $\phi \circ f_n \to \phi \circ f$ uniformly on E^c, so (since ϵ is arbitrary) $\phi \circ f_n \to \phi \circ f$ almost uniformly.

Finally suppose $f_n \to f$ in measure. Let $\epsilon > 0$, then let δ be as in $(*)$ above. Then

$$\mu(\{x : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| > \delta\}) \to 0 \quad \text{as } n \to \infty,$$

so $\phi \circ f_n \to \phi \circ f$ in measure.

(c) Let $\phi = \chi_{[0,\infty)}, f_n = -1/n, f = 0$ (i.e. f_n, f are constant functions.) Then $\phi \circ f_n = 0, \phi \circ f = 1$ so $f_n \to f$ in all 3 senses, but $\phi \circ f_n \to \phi \circ f$ in none of the 3 senses.

(38)(a) This follows from

$$\begin{aligned}
&\mu\Big(\{x : |(f_n + g_n)(x) - (f+g)(x)| > \epsilon\}\Big) \\
&\quad \leq \mu\Big(\{x : |(f_n(x) - f(x)| > \epsilon/2\}\Big) + \mu\Big(\{x : |(g_n(x) - g(x)| > \epsilon/2\}\Big) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

(b) Suppose $\mu(X) < \infty$. By problem (I), there exists M such that $\mu(\{x : |f_n(x)| > M\}) < \epsilon$ for all n and $\mu(\{x : |g(x)| > M\}) < \epsilon$. Now $|f_n g_n - fg| \leq |f_n(g_n - g)| + |(f_n - f)g|$, so

$$\begin{aligned}
&\mu(\{x : |(f_n g_n)(x) - (fg)(x)| > \epsilon\}) \\
&\quad \leq \mu(\{x : |f_n(x)| > M\}) + \mu(\{x : |g_n(x) - g(x)| > \frac{\epsilon}{2M}\}) \\
&\qquad + \mu(\{x : |f_n(x) - f(x)| > \frac{\epsilon}{2M}\}) + \mu(\{x : |g(x)| > M\}) \\
&\quad < \epsilon + \epsilon + \epsilon + \epsilon,
\end{aligned}$$

if n is large enough. Since ϵ is arbitrary, as in problem (35) this shows $f_n g_n \to fg$ in measure.

For a counterexample with $\mu(X) = \infty$, take $X = \mathbb{R}, \mu = m =$ Lebesgue, and $f = g$ unbounded, say $f(x) = g(x) = x^2$ on $\mathbb{R}$. Let $f_n = g_n = f + 1/n$. Then for $x \in [n, \infty)$ we have $|f_n(x)g_n(x) - f(x)g(x)| \geq 2x^2/n \geq 2n$, so $m(\{x : |f_n(x)g_n(x) - f(x)g(x)| \geq 1\}) \geq m([n,\infty)) \not\to 0$ as $n \to \infty$. Thus $f_n g_n \not\to fg$ in measure.

(42) Suppose $f_n \to f$ uniformly. Let $\epsilon > 0$. Then $\{x : |f_n(x) - f(x)| > \epsilon\} = \phi$ for n large enough, so $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$ as $n \to \infty$. Thus $f_n \to f$ in measure. (Note we do not need the fact that μ is counting measure for this part.)

Conversely suppose $f_n \to f$ in measure. Let $\epsilon > 0$. Then $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$, so $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < 1/2$ for n large enough. But since μ in counting measure, the only set of measure less than $1/2$ is ϕ. Therefore for n large enough, $|f_n(x) - f(x)| < \epsilon$ for all x. This means $f_n \to f$ uniformly.

(I)(a) Considering only integers $M \geq 1$, since f is real-valued we have $\cap_{M \geq 1}\{x : |f(x)| > M\} = \phi$. Since $\mu(X) < \infty$ we can use continuity from above to conclude that $\lim_{M \to \infty} \mu(\{x : |f(x)| > M\}) = 0$, so there exists M with $\mu(\{x : |f(x)| > M\}) < \epsilon$.

(b) Let $\epsilon > 0$. From (a), there exists M_0 such that $\mu(\{x : |f(x)| > M_0 - 1\}) < \epsilon$. By convergence in measure, there exists n_0 such that for $n \geq n_0$ we have $\mu(\{x : |f_n(x) - f(x)| >$

$1\}) < \epsilon$, and hence also

$$\mu(\{x : |f_n(x)| > M_0\}) \leq \mu(\{x : |f(x)| > M_0 - 1\}) + \mu(\{x : |f_n(x) - f(x)| > 1\}) < 2\epsilon.$$

By (a) again, for each $1 \leq n < n_0$ there exists M_n such that $\mu(\{x : |f_n(x)| > M_n\}) < \epsilon$. Letting $M = \max(M_0, M_1, .., M_{n_0-1})$ we then have $\mu(\{x : |f_n(x)| > M\}) < 2\epsilon$ for all $n \geq 1$. Since ϵ is arbitrary this completes the proof.

(II) Let $f \in L^1(m)$, let $\epsilon > 0$, let $M = \sup_x |g(x)|$ and let $\overline{g} = \int_0^1 g(x)\, dx$. By Theorem 2.26, there exists a step function $\varphi = \sum_{j=1}^k c_j \chi_{(a_j,b_j)}$ for which $\int |f - \varphi|\, dm < \epsilon$. For each j there is an open interval $I_j \subset (a_j, b_j)$ with endpoints that are multiples of $1/n$, and with these endpoints being within distance $1/n$ of a_j and b_j respectively. We have $\int_{I_j} (g(nx) - \overline{g})\, dx = 0$, since the integral over each period of length $1/n$ is 0, so

$$\begin{aligned}\left| \int_{\mathbb{R}} \chi_{(a_j,b_j)}(x)(g(nx) - \overline{g})\, dx \right| &= \left| \int_{(a_j,b_j)\backslash I_j} (g(nx) - \overline{g})\, dx \right| \\ &\leq 2M\mu((a_j, b_j)\backslash I_j) \\ &\leq \frac{4M}{n} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore also $\int_{\mathbb{R}} \varphi(x)(g(nx) - \overline{g})\, dx \to 0$, so for large n,

$$\begin{aligned}\left| \int_{\mathbb{R}} f(x)(g(nx) - \overline{g})\, dx \right| &\leq \left| \int_{\mathbb{R}} (f(x) - \varphi(x))(g(nx) - \overline{g})\, dx \right| + \left| \int_{\mathbb{R}} \varphi(x)(g(nx) - \overline{g})\, dx \right| \\ &\leq 2M \int_{\mathbb{R}} |f(x) - \varphi(x)|\, dx + \left| \int_{\mathbb{R}} \varphi(x)(g(nx) - \overline{g})\, dx \right| \\ &< 2M\epsilon + \epsilon.\end{aligned}$$

Since ϵ is arbitrary, this shows $\int_{\mathbb{R}} f(x)(g(nx) - \overline{g})\, dx \to 0$, which is equivalent to the desired result.

(III) Let $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$. Since $(1 + x^2)^n \geq 1 + nx^2 + \binom{n}{2}x^4$, we have $|f_n(x)| \leq 1$ for all $x \in [0, 1]$. Also, for $x > 0$, since $\binom{n}{2}$ is of order n^2, dividing numerator and denominator by n^2 shows that

$$0 \leq f_n(x) \leq \frac{1 + nx^2}{1 + nx^2 + \binom{n}{2}x^4} \to 0 \quad \text{as } n \to \infty.$$

Hence by Dominated Convergence, the desired limit is 0.

(IV)(a) Since f is not assumed to be in L^1, for $k \geq 1$ we need to define a truncated version $f_k = f\chi_{\{|f|\leq k\}}$. Let $F_k = \{x : |f(x)| > k\}$ be the set where f and f_k differ. Since $f_k \in L^1$, and continuous functions are dense in L^1, by Theorem 2.26 there exists a sequence of continuous functions which converges in L^1 to f_k. By Proposition 2.29, this sequence also converges in measure, and by Theorem 2.30 it has a subsequence which converges a.e. to f_k. By Egoroff's Theorem, this subsequence converges almost uniformly. Thus for each k, f_k is an almost-uniform limit of continuous functions. This means that given $k \geq 1$ and $\epsilon, \delta > 0$ we can find a continuous function g on $[a, b]$ and a set $E \subset [a, b]$ with $m(E) < \epsilon$, such that

$$\sup\{|f_k(x) - g(x)| : x \in [a, b]\backslash E\} < \delta,$$

and therefore, since $f = f_k$ on F_k^c,

$$\sup\{|f(x) - g(x)| : x \in [a, b]\backslash(E \cup F_k)\} < \delta.$$

Now since $F_k \searrow \phi$, for each $n \geq 1$ we can find a k_n such $m(F_{k_n}) < 1/2^n$. Applying the above with $\delta = 1/n$ and $\epsilon = 1/2^n$, we see that for each n there exist a continuous function g_n and a set E_n such that $m(E_n) < 1/2^n$ and

$$\sup\{|f(x) - g_n(x)| : x \in [a, b]\backslash(E_n \cup F_{k_n})\} < \frac{1}{n}.$$

For $j \geq 1$ let $\tilde{E}_j = \cup_{n \geq j}(E_n \cup F_{k_n})$; then $m(\tilde{E}_j) \leq \sum_{n \geq j} 2/2^n < 1/2^{j-2}$ and

$$\sup\{|f(x) - g_n(x)| : x \in [a, b]\backslash\tilde{E}_j\} < \frac{1}{n} \quad \text{for all } n \geq j,$$

which means that $g_n \to f$ uniformly on $[a, b]\backslash\tilde{E}_j$. Since $m(\tilde{E}_j) < 1/2^{j-2}$ and j is arbitrary, this says that $g_n \to f$ almost uniformly.

(b) By (a), there exist continuous functions g_n on $[a, b]$ such that $g_n \to f$ almost uniformly. Let $\epsilon > 0$; then there exists $\hat{E} \subset [a, b]$ with $m(\hat{E}^c) < \epsilon/2$ and $g_n \to f$ uniformly on $\hat{E}$. Since Lebesgue measure is regular, there exists a compact $E \subset \hat{E}$ with $m(\hat{E}\backslash E) < \epsilon/2$, so $m(E^c) \leq m(\hat{E}^c) + m(\hat{E}\backslash E) < \epsilon$. Since each $g_n|_E$ is continuous and the convergence is uniform on E, $f|_E$ is also continuous.