MATH 525a ASSIGNMENT 6
FALL 2016
Prof. Alexander
Due Monday October 24.

Note this assignment is due after the midterm, but this material is covered on the midterm. It may not be practical to do all problems before the midterm, though, in which case you might save the longer problems for after the midterm: (38), (II), and (IV)(a), and possibly also (37).

Folland Chapter 2 #33, 34, 35, 37, 38, 42 and:

(I)(a) Suppose $\mu(X) < \infty$ and f is a real-valued measurable function. Then given $\epsilon > 0$ there exists M such that $\mu(\{x : |f(x)| > M\}) < \epsilon$.

(b) Suppose $\mu(X) < \infty$ and f_n, f are real-valued with $f_n \to f$ in measure. Then part (a) can be done uniformly in n: given $\epsilon > 0$ there exists M such that $\mu(\{x : |f_n(x)| > M\}) < \epsilon$ for all n.

(II) Let $g : \mathbb{R} \to \mathbb{R}$ be bounded with period 1, and let m be Lebesgue measure. Show that for $f \in L^1(m)$,

$$\lim_n \int_{\mathbb{R}} f(x)g(nx)\ dx = \int_{\mathbb{R}} f(x)\ dx \int_0^1 g(x)\ dx.$$

This may be interpreted as a kind of averaging property: $g(nx)$ is a rapidly oscillating modification of g with period $1/n$, and in the limit, multiplying $f(x)$ by $g(nx)$ becomes the same as just multiplying by the average, which is $\int_0^1 g(x)\ dx$.

(III) Find

$$\lim_n \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n}\ dx.$$

(IV) Let f be a Lebesgue measurable function on an interval $[a, b]$.

(a) Show that there is a sequence $\{f_n\}$ of continuous functions with $f_n \to f$ almost uniformly on $[a, b]$.

(b) Given $\epsilon > 0$, show that there is a set E with $\mu(E^c) < \epsilon$ such that the restriction of f to E is continuous. In fact one can take E to be compact. This is Lusin's Theorem.

HINTS:

(33) Use 2.30.

(34)(a) If this were false, it would mean there was a subsequence $\{f_{n_k}\}$ and an $\epsilon > 0$ such that $|\int f_{n_k} - \int f| > \epsilon$ for all k.

(35) Essentially from the definitions, one can say that to get convergence in measure, it's enough to show that for every $\epsilon, \delta > 0$ there exists N such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \delta$ for all $n \geq N$. The point is that it is enough to establish the inequality with $\epsilon = \delta$.

(38)(b) Problem (I) may be useful. Also, use $|f_n g_n - fg| \leq |f_n g_n - f_n g| + |f_n g - fg|$. For an example with $\mu(X) = \infty$, try $f_n = f + \frac{1}{n}$.

(I)(b) First apply part (a) to f and use this to help find an M that works for all sufficiently large n. Then deal with the finite number of n that are not "sufficiently large."

(II) This result is simpler for a special type of function f. Use Theorem 2.26 to approximate f by a function of this special type. This problem really illustrates the value of results like Theorem 2.26!

(IV)(a) This is somewhat tricky; assume it and do (b) if you have to. Use the last part of Theorem 2.26. A complication: we do not assume $f \in L^1$, as required in 2.26. So consider truncated functions, that is, functions $f_k = f\chi_{\{|f| \leq k\}}$ with k large.

(b) Use (a) and Theorem 1.18. This theorem enables you to consider continuous functions converging uniformly on a compact set, instead of a general measurable set; why is this useful?