MATH 525a ASSIGNMENT 5 SOLUTIONS
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Prof. Alexander

Chapter 2

(20) We may assume all the functions are real-valued, otherwise consider real and imaginary parts separately. Then $g_n + f_n \geq 0$ and $g_n - f_n \geq 0$, so by Fatou's Lemma,

$$\int g + \liminf \int f_n = \lim \int g_n + \liminf \int f_n = \liminf \int (g_n + f_n) \geq \int (g + f),$$

$$\int g - \limsup \int f_n = \lim \int g_n - \limsup \int f_n = \liminf \int (g_n - f_n) \geq \int (g - f).$$

Subtracting $\int g$ from both sides we get $\liminf \int f_n \geq \int f \geq \limsup \int f_n$, so $\int f_n \to \int f$.

(21) If $\int |f_n - f| \to 0$ then

$$\left| \int |f_n| - \int |f| \right| \leq \int \big| |f_n| - |f| \big| \leq \int |f_n - f| \to 0.$$

Conversely suppose $\int |f_n| \to \int |f|$. Let $g_n = |f_n| + |f|, g = 2|f|, h_n = |f_n - f|, h = 0$. Then $h_n, g_n, h, g \in L^1, h_n \to h$ a.e., $g_n \to g$ a.e., $|h_n| \leq g_n, \int g_n \to \int g$, so by problem 20, $\int h_n \to \int h$, that is, $\int |f_n - f| \to 0$.

(22) If f is a function on $\mathbb{N}$, then $\int f$ is well-defined provided the positive and negative terms in the sum $\sum_{k=1}^{\infty} f(k)$ are not both infinite, and in this case, $\int f \; d\mu$ is equal to this sum. Fatou's Lemma says that if $f_n(k) \geq 0$ for all n, k, then

$$\sum_{k=1}^{\infty} \liminf_n f_n(k) \geq \liminf_n \sum_{k=1}^{\infty} f_n(k).$$

Monotone Convergence says that if $0 \leq f_n(k) \nearrow f(k)$ for all k, then

$$\sum_{k=1}^{\infty} f(k) = \lim_n \sum_{k=1}^{\infty} f_n(k).$$

Dominated Convergence says that if there exists a sequence $g(k) \geq 0$ with $\sum_k g(k) < \infty$ and $|f_n(k)| \leq g(k)$ for all n, k, then $\sum_{k=1}^{\infty} f(k)$ converges and

$$\sum_{k=1}^{\infty} f(k) = \lim_n \sum_{k=1}^{\infty} f_n(k).$$

(25)(a) From calculus, $\int f\ dm = \int_0^1 x^{-1/2}\ dx = 2x^{1/2}|_0^1 = 2$. Let $g_k(x) = \sum_{n=1}^k 2^{-n} f(x - r_n)$. Since f is nonnegative, $g_k \nearrow g$, so by Monotone Convergence we have

$$\int g\ dm = \lim_k \int g_k\ dm = \lim_k \sum_{n=1}^k 2^{-n} \int f(x - r_n)\ m(dx) = 2 \lim_k \sum_{n=1}^k 2^{-n} = 2 < \infty,$$

so $g < \infty$ a.e.

(b) Given a nonempty interval (a,b) there exists a rational $r_n \in (a,b)$. Then $g(x) \geq 2^{-n} f(x - r_n) \nearrow \infty$ as $x \searrow r_n$, so g is unbounded on every interval. This means g cannot be continuous anywhere, since if it were continuous at some x it would be bounded in a neighborhood of x. Even if g is modified on a null set N, for each rational r_n there is a sequence $x_k \notin N$ with $x_k \searrow r_n$, so $g(x_k) \to \infty$ as above, so g is still unbounded on every interval, and hence continuous nowhere.

(c) Since $g < \infty$ a.e., we have $g^2 < \infty$ a.e. If (a,b) is an interval and r_n is a rational in (a,b), then for some $\epsilon > 0$,

$$\int_{(a,b)} g^2\ dm \geq \int_a^b 2^{-2n} f^2(x - r_n)\ dx \geq 2^{-2n} \int_{r_n}^{r_n + \epsilon} f^2(x - r_n)\ dx$$

$$= 2^{-2n} \int_0^\epsilon f^2(u)\ du = 2^{-2n} \int_0^\epsilon u^{-1}\ du = \infty.$$

(I) Let $t_n \searrow 0$. Then

$$\frac{F(t_n) - F(0)}{t_n - 0} = \int_{(0,\infty)} \frac{1}{t_n} \left(\frac{1}{x + t_n} - \frac{1}{x} \right)\ \mu(dx) = -\int_{(0,\infty)} \frac{1}{x(x + t_n)}\ \mu(dx).$$

Let $f_n(x) = 1/(x(x+t_n))$ and $f(x) = 1/x^2$. Then $0 \leq f_n \nearrow f$ so by Monotone Convergence,

$$\lim_n \frac{F(t_n) - F(0)}{t_n - 0} = -\lim_n \int_{(0,\infty)} f_n\ d\mu = -\int_{(0,\infty)} f\ d\mu = -\int_{(0,\infty)} \frac{1}{x^2}\ \mu(dx),$$

so

$$F^R(0) = -\int_{(0,\infty)} \frac{1}{x^2}\ \mu(dx).$$

(II) Let

$$f(x,u) = \frac{x^n e^{ux}}{e^x + 1}.$$

Then

$$\frac{\partial f}{\partial u}(x,u) = \frac{x^{n+1} e^{ux}}{e^x + 1}.$$

In differentiating g at a point $u_0 \in (0,1)$ we can restrict u to an interval $[a,b] \subset (0,1)$ containing u_0 in its interior, that is, $0 < a < u_0 < b < 1$. For all $u \in [a,b]$ and $x \in \mathbb{R}$ we have

$$\left|\frac{\partial f}{\partial u}(x,u)\right| \le G(x) = \begin{cases} |x|^{n+1}e^{ux} \le |x|^{n+1}e^{-a|x|}, & x < 0, \\ |x|^{n+1}e^{(u-1)x} \le |x|^{n+1}e^{-(1-b)|x|}, & x \ge 0. \end{cases}$$

Since a and $1-b$ are positive, G is integrable on $\mathbb{R}$. By Theorem 2.27, g is differentiable.

(III)(a) If $E = (a,b]$ is an h-interval, then $m(E/c) = m((b/c, a/c]) = a/c - b/c = m(E)/c$, so $\nu(E) = m(E)$. Since m and ν are Lebesgue-Stieltjes measures that agree on h-intervals, by uniqueness they are the same.

(b) For an indicator $f = \chi_E$ we have $c\int \chi_E(cx)\ m(dx) = c\int \chi_{E/c}(x)\ m(dx) = \nu(E) = m(E) = \int \chi_E(x)\ m(dx)$ by part (a), so the result is true when f is such an indicator, and hence also (by linearity) when f is a simple function. For general $f \in L^+$ there are simple functions $\varphi_n \nearrow f$, so they satisfy $c\int \varphi_n(cx)\ m(dx) = \int \varphi_n(x)\ m(dx)$. By Monotone Convergence we can take the limit of both the left and right sides to obtain $c\int f(cx)\ m(dx) = \int f(x)\ m(dx)$.

(c) Let $g(x) = \sum_{n=1}^{\infty} |f(nx)|/n^\gamma$. By part (b) and Monotone Convergence we have

$$\int g(x)\ m(dx) = \sum_{n=1}^{\infty} \frac{1}{n^\gamma} \int |f(nx)|\ m(dx) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\gamma}} \int |f(x)|\ m(dx) < \infty.$$

This shows that $g < \infty$ a.e. For any x with $g(x) < \infty$, the terms of the series must approach 0, so we conclude $f(nx)/n^\gamma \to 0$ a.e.

(IV)(a) Since f is strictly positive, we have $\phi = \cap_{n\ge 1}\{x : f(x) \le 1/n\}$. Hence $\lim_n \mu(\{x : f(x) \le 1/n\} = 0$ so there exists N with $\mu(\{x : f(x) \le 1/N\}) < \alpha/2$. If E satisfies $\mu(E) > \alpha$ this means

$$\mu(E \cap \{x : f(x) > 1/N\}) \ge \mu(E) - \mu(\{x : f(x) \le 1/N\}) > \alpha/2$$

and therefore

$$\int_E f\ d\mu \ge \int_{E\cap\{x:f(x)>1/N\}} f\ d\mu \ge \frac{1}{N}\mu(E \cap \{x : f(x) > 1/N\}) > \frac{1}{N}\frac{\alpha}{2}.$$

This shows the desired infimum over E is strictly positive.

(b) Let $X = (0,\infty), f(x) = 1/x$ and let μ be Lebesgue measure. Then f is strictly positive, but given $\alpha > 0$ we have

$$\lim_n \int_{[n,n+\alpha]} f\ d\mu = \lim_n \log\left(1 + \frac{\alpha}{n}\right) = 0,$$

so the desired infimum is 0.

(V) Let $f_n(x) = \frac{x}{n} f(x) \chi_{[0,n]}(x)$. Then $0 \leq f_n(x) \leq f(x)$, $f_n(x) \to f(x) = 0$ for all x, and $f \in L^1$, so by Dominated Convergence,

$$\lim_n \frac{1}{n} \int_{[0,n]} x f(x)\ dx = \lim_n \int f_n(x)\ dx = \int f(x)\ dx = 0.$$