MATH 525a ASSIGNMENT 5
FALL 2016
Prof. Alexander
Due Friday October 14.

Folland Chapter 2 #20, 21, 22, 25 and:

(I) The right derivative of a function F at a point t_0 is defined to be

$$F^R(t_0) = \lim_{t \searrow t_0} \frac{F(t) - F(t_0)}{t - t_0},$$

whenever this limit exists. (The value of the limit is allowed to be $\pm\infty$.)

Let μ be a measure on $(0, \infty)$ with $\int_{(0,\infty)} \frac{1}{x}\ \mu(dx) < \infty$, and let $F(t) = \int_{(0,\infty)} \frac{1}{x+t}\ \mu(dx)$ for $t \geq 0$. Show that

$$F^R(0) = -\int_{(0,\infty)} \frac{1}{x^2}\ \mu(dx)$$

(even if this value is $-\infty$.)

(II) Let $n \geq 1$. Show that the function

$$g(u) = \int_{-\infty}^{\infty} \frac{x^n e^{ux}}{e^x + 1}\ dx, \qquad u \in (0, 1),$$

is differentiable in $(0, 1)$.

(III)(a) Let $c > 0$, let m be Lebegue measure, and define $\nu(E) = cm(E/c)$, where $E/c = \{x/c : x \in E\}$. It is easily checked that ν is a measure (you need not do this.) Show that $\nu = m$.

(b) Let $f \in L^1(\mathbb{R})$ and $c > 0$. Show that $\int f(cx)\ m(dx) = \frac{1}{c} \int f(x)\ m(dx)$.

(c) Let $f \in L^1(\mathbb{R})$ and $\gamma > 0$, and let $f_n(x) = f(nx)/n^\gamma,\ n \geq 1$. Show that $f_n \to 0$ a.e.

(IV) Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $f \in L^1(\mu)$ is strictly positive. Let $0 < \alpha < \mu(X)$.

(a) Show that

$$\inf \left\{ \int_E f\ d\mu : \mu(E) \geq \alpha \right\} > 0.$$

(b) Show by example that (a) can be false if we remove the assumption $\mu(X) < \infty$.

(V) Let $f \geq 0$ and suppose $f \in L^1([0,\infty))$. Show that

$$\lim_n \frac{1}{n} \int_0^n x f(x)\, dx = 0.$$

HINTS:

(21) One direction uses Exercise 20, the other uses a direct comparison of $\int |f_n - f|$ to the difference between $\int |f_n|$ and $\int |f|$.

(25)(b) If you show g is unbounded in every interval, this implies g is discontinuous at every point (say why this is so.)

(c) Find a lower bound for g^2 which is not integrable.

(I) You cannot plug into the text theorem on differentiating under the integral sign, as the Dominated Convergence Theorem does not apply here. It is enough to consider sequences $t_n \searrow 0$.

(II) You will have difficulty if you try to do things uniformly over $u \in (0,1)$. But you don't need to do things uniformly over this whole interval.

(III)(a) What sets do you need to show ν and m agree on?

(b) For a certain type of function, this is easy to prove, and you can extend from there to general functions.

(c) Construct a function g using the f_n's, such that finiteness of $g(x)$ ensures the desired convergence at x. What would ensure that your g is finite a.e.?

(IV)(a) Consider $\{x : f(x) \geq 1/n\}$ for large n.