MATH 525a ASSIGNMENT 4 SOLUTIONS
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Chapter 2

(3) Since f_n is measurable for all n, so is $g = \limsup f_n - \liminf f_n$, by 2.6 and 2.7. Hence $\{x : \lim_n f_n(x) \text{ exists}\} = g^{-1}(0)$ is a measurable set.

(6) Let E be a non-lebesgue-measurable subset of $\mathbb{R}$. Then $\sup_{a\in E} \chi_{\{a\}} = \chi_E$ is not a measurable function, though $\chi_{\{a\}}$ is measurable for each $a \in E$.

(13) Using Fatou's Lemma twice,

$$\begin{aligned}
\int_E f &\le \liminf \int_E f_n \\
&\le \limsup \int_E f_n \\
&= \limsup \left(\int f_n - \int_{E^c} f_n \right) \\
&= \lim_n \int f_n - \liminf \int_{E^c} f_n \\
&\le \int f - \int_{E^c} \liminf f_n \\
&= \int f - \int_{E^c} f \\
&= \int_E f.
\end{aligned}$$

Therefore all of these are equal, in particular the right sides of the first two inequalities are equal to each other, and to $\int_E f$, which says that $\lim_n \int_E f_n = \int_E f$.

For an example let $f_n = n\chi_{(0,1/n)} + \chi_{[1,n]}$ and $f = \chi_{[1,\infty)}$. Then $f_n \to f$ pointwise and $\int f = \lim_n \int f_n = \infty$, but $\int_{(0,1)} f_n = 1 \not\to \int_{(0,1)} f = 0$.

(14) Let $f \in L^+$ and $\lambda(E) = \int_E f \; d\mu$. Clearly $\lambda(\phi) = 0$. For $E_1, E_2, ... \in \mathcal{M}$ disjoint, we have

$$0 \le f\chi_{\cup_1^n E_i} \nearrow f\chi_{\cup_1^\infty E_i},$$

so by Monotone Convergence,

$$\lambda\left(\cup_1^\infty E_i\right) = \int f\chi_{(\cup_1^\infty E_i)} = \lim_n \int f\chi_{(\cup_1^n E_i)} = \lim_n \int \sum_{i=1}^n f\chi_{E_i} = \lim_n \sum_{i=1}^n \int_{E_i} f = \sum_{i=1}^\infty \lambda(E_i),$$

meaning λ is countably additive.

Next, for simple $g = \sum_{i=1}^m c_i\chi_{F_i}$ in L^+, we have

$$\int g \; d\lambda = \sum_{i=1}^m c_i\lambda(F_i) = \sum_{i=1}^m c_i \int_{F_i} f \; d\mu = \int \sum_{i=1}^m c_i f\chi_{F_i} \; d\mu = \int fg \; d\mu.$$

For general $g \in L^+$, let $0 \le \varphi_n \nearrow g$ with φ_n simple. Then by Monotone Convergence (twice), since $f\varphi_n \nearrow fg$,

$$\int g \; d\lambda = \lim_n \int \varphi_n \; d\lambda = \lim_n \int f\varphi_n \; d\mu = \int fg \; d\mu.$$

(15) We have $0 \le f_1 - f_n \nearrow f_1 - f$, so by Monotone Convergence, $\lim_n \int(f_1 - f_n) = \int(f_1 - f)$. Subtracting $\int f_1$ from both sides and taking the negative gives $\lim_n \int f_n = \int f$.

(16) Let $E_n = \{x : f(x) \ge 1/n\}$. Then $f \ge \frac{1}{n}\chi_{E_n}$ so for all n, $\frac{1}{n}\mu(E_n) = \int \frac{1}{n}\chi_{E_n} \le \int f < \infty$, meaning $\mu(E_n) < \infty$. Now $E_1 \subset E_2 \subset ...$ and $\cup_n E_n = \{x : f(x) > 0\}$, so by continuity from below for the measure $\lambda(A) = \int_A f$, we get

$$\int_{E_n} f = \lambda(E_n) \to \lambda(\cup_i E_i) = \int_{(\cup_i E_i)} f = \int f.$$

Thus there exists n satisfying both $\mu(E_n) < \infty$ and $\int_{E_n} f > \int f - \epsilon$.

(A)(a) Let $\epsilon > 0$. We first approximate $|f|$ by a bounded function: let $E_n = \{x : |f(x)| \le n\}$ and $f_n = |f|\chi_{E_n}$. By Monotone Convergence, $\int f_n \nearrow \int |f|$, so there exists N such that

$$\int_X (|f| - f_N) \; d\mu = \int_X |f| \; d\mu - \int_X f_N \; d\mu < \frac{\epsilon}{2}.$$

Since $f_N \le N$, we then have

$$\mu(A) < \frac{\epsilon}{2N} \implies \int_A f_N \; d\mu \le \int_A N \; d\mu = N\mu(A) < \frac{\epsilon}{2}.$$

Therefore

$$\mu(A) < \frac{\epsilon}{2N} \implies \int_A |f| \; d\mu = \int_A f_N \; d\mu + \int_A (|f| - f_N) \; d\mu < \frac{\epsilon}{2} + \int_X (|f| - f_N) \; d\mu < \epsilon.$$

(b) Let $\epsilon > 0$. By (a), there exists $\delta > 0$ such that

$$0 < y - x < \delta \implies m((x,y]) < \delta \implies |F(y) - F(x)| = \left| \int_{(x,y]} f \; dm \right| \le \int_{(x,y]} |f| \; dm < \epsilon.$$

This shows F is continuous (in fact uniformly continuous.)

(B) No. An example from lecture shows this: let $f_n = n\chi_{(0,1/n]}$. Then $f_n \to 0$ a.e., so $\int \limsup_n f_n = 0$, but $\int f_n = 1$ for all n so $\limsup_n \int f_n = 1$.

(C) SOLUTION 1, BASED ON THE HINT: Let $\epsilon_k \searrow 0$. We have

$$\begin{aligned} f_n(x) \not\to 0 &\iff \text{for some } \epsilon > 0,\ |f_n(x)| > \epsilon \text{ for infinitely many } n \\ &\iff \text{for some } k,\ |f_n(x)| > \epsilon_k \text{ for infinitely many } n \\ &\iff \text{for some } k,\ \text{we have } x \in \cap_m B_m(\epsilon_k). \end{aligned} \tag{1}$$

Now since the given series converges, its tail converges to 0, so that for fixed ϵ,

$$\mu(B_m(\epsilon)) = \mu\left(\cup_{n \ge m}\{x : |f_n(x)| > \epsilon\}\right) \le \sum_{n=m}^{\infty} \mu\left(\{x : |f_n(x)| > \epsilon\}\right) \to 0 \text{ as } m \to \infty,$$

so $\mu(\cap_m B_m(\epsilon)) = 0$. Therefore by (1),

$$\mu\left(\{x : f_n(x) \not\to 0\}\right) \le \sum_{k=1}^{\infty} \mu\left(\cap_m B_m(\epsilon_k)\right) = 0,$$

that is, $f_n \to 0$ a.e.

SOLUTION 2: Fix $\epsilon > 0$ and let $A_n = \{x : |f_n(x)| > \epsilon\}$. Then

$$\sum_n \chi_{A_n}(x) < \infty \implies x \in A_n \text{ for only finitely many } n \implies \limsup_n |f_n(x)| \le \epsilon.$$

Further, by assumption we have $\int \left(\sum_n \chi_{A_n}\right) \, d\mu = \sum_n \mu(A_n) < \infty$. Therefore $\sum_n \chi_{A_n} < \infty$ a.e., so $\limsup_n |f_n(x)| \le \epsilon$ for almost every x. Since ϵ is arbitrary, this shows $f_n \to 0$ a.e.

(D)(a) Let $g = \sum_{j=1}^{7} \chi_{F_j}$. Then $\int_X g \; d\mu = \sum_{j=1}^{7} \mu(F_j) \ge 7/2 > 3$. There must then be an $x \in X$ with $g(x) > 3$, since otherwise we would have $\int_X g \; d\mu \le \int_X 3 \; d\mu = 3$. But then $g(x) \ge 4$, which means x is in at least 4 of the sets F_j.