MATH 525a ASSIGNMENT 4
FALL 2016
Prof. Alexander
Due Wednesday October 5

Folland Chapter 2 #3, 6, 13, 14, 15, 16 plus these problems:

(A)(a) Let $(X, \mathcal{M}, \mu)$ be a measure space and f an integrable function. Show that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\mu(A) < \delta \implies \int_A |f| \, d\mu < \epsilon.$$

(b) For Lebesgue measure m, suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $a \in \mathbb{R}$. Define $F(x) = \int_a^x f \, dm$. Show that F is continuous.

(B) For $f_n \geq 0$ Lebesgue measurable functions on $\mathbb{R}$, is it necessarily true that

$$\limsup_n \int f_n \leq \int \limsup_n f_n \ ?$$

If not, give a counterexample.

(C) Suppose $\{f_n\}$ are functions on $(X, \mathcal{M}, \mu)$ and $\sum_n \mu(\{x : |f_n(x)| > \epsilon\}) < \infty$ for all $\epsilon > 0$. Show that $f_n \to 0$ a.e.

(D) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, and suppose $F_1, \ldots, F_7$ are 7 measurable sets with $\mu(F_j) \geq 1/2$ for all j.

(a) Show that there exist indices $i_1 < i_2 < i_3 < i_4$ for which $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \phi$.

(b) Would (a) be correct if we started with 6 measurable sets instead of 7?

HINTS:

(3) This is very short if you approach it right. Relate the limit to lim sup and lim inf, and use Prop. 2.7.

(6) You can make an example using indicator functions.

(13) Use Fatou's Lemma. To get an example where the conclusion fails without the finiteness assumption, define f_n, f on $[0,1]$ so that $\int_{[0,1]} f_n \not\to \int_{[0,1]} f$, then extend the definitions of f_n, f to all of $\mathbb{R}$ in such a way that the required conditions are satisfied.

(14) Consider g simple, then use Monotone Convergence.

(15) See the proof of Monotone Convergence.

(A)(a) This is easy for a bounded function. How can you use this to prove the general case?

(C) This is a bit tricky. Relate the set $\{x : f_n(x) \not\to 0\}$ to the sets

$$B_m(\epsilon) = \{x : |f_n(x)| > \epsilon \text{ for some } n \geq m\}.$$

What does the assumption in the problem tell you about the values $\mu(B_m(\epsilon))$?

(D) Restate the problem as one about (sums of) indicator functions, and their integrals. For a point in the intersection in (a), what is true about the indicators and/or their sum?