MATH 525a ASSIGNMENT 3 SOLUTIONS
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Chapter 1

(18)(a) By the definition of μ^*, there exist $\{A_j, j \geq 1\}$ with $A_j \in \mathcal{A}, E \subset \cup_1^\infty A_j$, and $\sum_{j=1}^\infty \mu(A_j) \leq \mu^*(E) + \epsilon$. Let $A = \cup_1^\infty A_j$. Then $A \in \mathcal{A}_\sigma, E \subset A$ and $\mu^*(A) \leq \sum_1^\infty \mu^*(A_j) = \sum_1^\infty \mu(A_j) \leq \mu^*(E) + \epsilon$.

(b) Suppose first that E is μ^*-measurable. By (a), for each $n \geq 1$ there exists $B_n \in \mathcal{A}_\sigma$ with $E \subset B_n, \mu^*(B_n) \leq \mu^*(E) + 1/n$. Let $B = \cap_n B_n$. Then $E \subset B$ and $B \in \mathcal{A}_{\sigma\delta}$ and $\mu^*(E) \leq \mu^*(B) \leq \mu^*(B_n) \leq \mu^*(E) + 1/n$ for all n, so $\mu^*(E) = \mu^*(B)$. Since E is μ^*-measurable, we have $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \backslash E) = \mu^*(E) + \mu^*(B \backslash E) = \mu^*(B) + \mu^*(B \backslash E)$, so $\mu^*(B \backslash E) = 0$.

Conversely suppose there exists $B \in \mathcal{A}_{\sigma\delta}$ with $\mu^*(B \backslash E) = 0$. By the preceding paragraph (applied to $B \backslash E$ in place of E), there exists $N \in \mathcal{A}_{\sigma\delta}$ with $B \backslash E \subset N$ and $\mu^*(N) = 0$. Since $N \in \mathcal{A}_{\sigma\delta}$ and the μ^*-measurable sets form a σ-algebra, we can conclude that N is μ^*-measurable. Since μ^* is a complete measure on the σ-algebra {all μ^*-measurable sets}, this shows that $B \backslash E$ is μ^*-measurable. Hence $E^c = B^c \cup (B \backslash E)$ is a union of two μ^*-measurable sets so is μ^*-measurable, and therefore E is μ^*-measurable as well.

(26) Let $E \in \mathcal{M}_\mu$ with $\mu(E) < \infty$, and $\epsilon > 0$. By Theorem 1.18, there is an open $U \supset E$ with $\mu(U \backslash E) < \epsilon/2$. Since U is open in $\mathbb{R}$, U is a finite or countable union of disjoint open intervals: $U = \cup_j I_j$. If this is a finite union, we are done, so suppose it is infinite. There exists n such that for $A = \cup_{j=1}^n I_j$,

$$\mu(A) = \sum_{j=1}^n \mu(I_j) > \left(\sum_{j=1}^\infty \mu(I_j)\right) - \frac{\epsilon}{2} = \mu(U) - \frac{\epsilon}{2}.$$

Therefore $\mu(E \triangle A) = \mu(E \backslash A) + \mu(A \backslash E) \leq \mu(U \backslash A) + \mu(U \backslash E) < \epsilon/2 + \epsilon/2 = \epsilon$.

(28) Let $a_n < a$ with $a_n \nearrow a$. Then by continuity from above, $\mu_F(\{a\}) = \mu_F(\cap_n (a_n, a]) = \lim_n \mu_F((a_n, a]) = \lim_n (F(a) - F(a_n)) = F(a) - F(a-)$.

Then using this result we get $\mu_F([a, b)) = \mu_F((a, b]) + \mu_F(\{a\}) - \mu_F(\{b\}) = F(b) - F(a) + F(a) - F(a-) - (F(b) - F(b-)) = F(b-) - F(a-)$.

Similarly, $\mu_F([a, b]) = \mu_F((a, b]) + \mu_F(\{a\}) = F(b) - F(a) + F(a) - F(a-) = F(b) - F(a-)$, and $\mu_F((a, b)) = \mu_F((a, b]) - \mu_F(\{b\}) = F(b) - F(a) - F(b) + F(b-) = F(b-) - F(a)$.

(30) Let $E \in \mathcal{L}$ with $m(E) > 0$ and let $\alpha \in (0, 1)$. By Theorem 1.18, there is an open $U \supset E$ with $m(U) < m(E)/\alpha$. Since U is open in $\mathbb{R}$, U is a finite or countable union of disjoint

open intervals: $U = \cup_j I_j$. Therefore

$$\sum_j m(E \cap I_j) = m(E) > \alpha m(U) = \alpha \sum_j m(I_j).$$

This means there must exist at least one j with $m(E \cap I_j) > \alpha m(I_j)$, so we can take $I = I_j$.

(31) Let $E \in \mathcal{L}$ with $m(E) > 0$, and let $\alpha \in (3/4, 1)$. By #30 there exists an interval I with $m(E \cap I) > \alpha m(I)$. Let $J = (-\frac{1}{2}m(I), \frac{1}{2}m(I))$ and let $F = E \cap I$. Suppose there is a $z \in J$ with $z \notin F - F$. This means that F and its translate $F_z = \{x + z : x \in F\}$ are disjoint. (If there were a $y \in F \cap F_z$, we would have $y = x + z$ for some $x \in F$ and then $z = y - x \in F - F$, a contradiction.) Let $I_z = \{x + z : x \in I\}$. Then $F \cup F_z$ is contained in $I \cup I_z$, which is an interval of length at most $\frac{3}{2}m(I)$, since $|z| < \frac{1}{2}m(I)$. Hence

$$2\alpha m(I) < 2m(F) = m(F \cup F_z) \le m(I \cup I_z) \le \frac{3}{2}m(I),$$

which gives $\alpha < 3/4$, a contradiction. Thus no such z exists, i.e. $J \subset F - F \subset E - E$.

(I) We need to verify countable additivity. Suppose $E_1, E_2, \dots$ are disjoint. By subadditivity of outer measures we have $\mu^*(\cup_{n=1}^\infty E_n) \le \sum_{n=1}^\infty \mu^*(E_n)$. By monotonicity and finite additivity, for all N we have

$$\mu^*(\cup_{n=1}^\infty E_n) \ge \mu^*(\cup_{n=1}^N E_n) \ge \sum_{n=1}^N \mu^*(E_n).$$

Letting $N \to \infty$ shows that $\mu^*(\cup_{n=1}^\infty E_n) \ge \sum_{n=1}^\infty \mu^*(E_n)$, so we have equality, i.e. countable additivity.

(II) Let $E_N = E \cap [-N, N]$. We will show that $m(\{x^2 : x \in E_N\}) = 0$ for all N, which is sufficient since $\{x^2 : x \in E\} = \cup_{N=1}^\infty \{x^2 : x \in E_N\}$. For an interval $(a, b) \subset [0, N+1]$, we have $m(\{x^2 : x \in (a, b)\}) = b^2 - a^2 = (b + a)(b - a) \le 2(N + 1)m((a, b))$. It is easily checked that similarly,

$$m(\{x^2 : x \in (a, b)\}) \le 2(N + 1)m((a, b))$$

for all $(a, b) \subset [-N - 1, N + 1]$. Let $\epsilon > 0$. Now

$$0 = m(E_N) = \inf \left\{ \sum_{j=1}^\infty m((a_j, b_j)) : E_N \subset \cup_{j=1}^\infty (a_j, b_j) \right\}$$

and we can restrict the infimum to intervals contained in $[-N-1, N+1]$, there exist intervals $(a_j, b_j) \subset [-N - 1, N + 1]$ with $E_N \subset \cup_{j=1}^\infty (a_j, b_j)$ and $\sum_{j=1}^\infty m((a_j, b_j)) < \epsilon/2(N + 1)$. Then

$$m(\{x^2 : x \in E_N\}) \le \sum_{j=1}^\infty m\left(\{x^2 : x \in (a_j, b_j)\right) \le 2(N + 1) \sum_{j=1}^\infty m((a_j, b_j)) < \epsilon,$$

and ϵ is arbitrary, so $m(\{x^2 : x \in E_N\}) = 0$.

(III)(a) Let $G = \cup_{\alpha\in A} N_\alpha$ be a union of open null sets. Let us refer to a ball (i.e. interval) with rational center and radius as a *rational ball.* There are only countably many rational balls in $\mathbb{R}$. If a rational ball is contained in at least one N_α, let us call it an *included rational ball.* For each included rational ball B, we can pick a particular index $\alpha(B)$ such that $B \subset N_{\alpha(B)}$ (by the Axiom of Choice.)

For each $x \in G$ there is an included rational ball B_x with $x \in B_x \subset N_{\alpha(B_x)}$. It follows that G is actually the countable union $G = \cup_B N_{\alpha(B)}$, where the union is over all included rational balls. Therefore G is null.

(b) Suppose $F(x') = F(x'')$ for some $x' < x < x''$. Then $\mu((x', x'')) \le F(x'') - F(x') = 0$ so x is contained in the open null set (x', x''), so $x \notin \text{Supp}(\mu)$. Conversely suppose $x \notin \text{Supp}(\mu)$. Then there is a null open set N containing x; this open set contains an interval $(x', y) \ni x$ and for every $x'' \in (x, y)$ we have $F(x'') - F(x') = \mu((x', x'']) \le \mu((x', y)) \le \mu(N) = 0$.

(IV) For $n = 2$ it is an equality (see Ch. 1 #9): $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2)$. Using induction, suppose that for some $k \ge 2$ the inquality is valid whenever $n \le k$, and consider measurable sets $A_1, \ldots, A_{k+1}$. Using the cases $n = 2$ and $n = k$ we have

$$\begin{aligned}
\mu\left(\cup_{i=1}^{k+1} A_i\right) &= \mu\left(\cup_{i=1}^{k} A_i\right) + \mu(A_{k+1}) - \mu\left(\left(\cup_{i=1}^{k} A_i\right) \cap A_{k+1}\right) \\
&\ge \sum_{i=1}^{k} \mu(A_i) - \sum_{1\le i<j\le k} \mu(A_i \cap A_j) + \mu(A_{k+1}) - \mu\left(\cup_{i=1}^{k} \left(A_i \cap A_{k+1}\right)\right) \\
&\ge \sum_{i=1}^{k+1} \mu(A_i) - \sum_{1\le i<j\le k} \mu(A_i \cap A_j) - \sum_{i=1}^{k} \mu\left(A_i \cap A_{k+1}\right) \\
&= \sum_{i=1}^{k+1} \mu(A_i) - \sum_{1\le i<j\le k+1} \mu(A_i \cap A_j),
\end{aligned}$$

so the desired inequality is also valid for $n = k + 1$. Thus it is valid for all n.