MATH 525a ASSIGNMENT 3
FALL 2016
Prof. Alexander
Due Friday September 23

Folland Chapter 1 #18ab, 26, 28 ($[a, b]$ only), 30, 31, and the following problems:

(I) Suppose μ^* is an outer measure which is finitely additive. Show that μ^* is actually a measure.

(II) Suppose $E \subset \mathbb{R}$ has Lebesgue measure 0. Show that $\{x^2 : x \in E\}$ also has Lebesgue measure 0.

(III)(a) Let μ be a measure on the Borel sets $\mathcal{B}_\mathbb{R}$ in $\mathbb{R}$. Show that an arbitrary (not necessarily countable) union of open null sets is null. If we let U be the union of all open null sets, it follows that U^c is the smallest closed set whose complement is null. U^c is called the *support* of μ, denoted $\mathrm{Supp}(\mu)$.

(b) Suppose μ is finite on bounded sets and let F be its distriubution function. Show that $x \in \mathrm{Supp}(\mu)$ if and only if $F(x') < F(x'')$ for all $x' < x < x''$.

(IV) Let μ be a finite measure on $(X, \mathcal{M})$. Show that

$$\mu\left(\cup_{i=1}^n A_i\right) \geq \sum_{i=1}^n \mu(A_i) - \sum_{1 \leq i < j \leq n} \mu(A_i \cap A_j)$$

for all $n \geq 2$ and $A_1, \ldots, A_n \in \mathcal{M}$.

HINTS:

(18)(b) To get "if", show that $\mu^*(B \backslash E) = 0$ implies $B \backslash E$ is μ^*-measurable.

(30) First prove it for an open set in place of the open interval I. Use Theorem 1.18. An open set in $\mathbb{R}$ consists of at most countably many open intervals.

(31) This one is rather tricky, but at least give it a try. Let $F = E \cap I$ and proceed by contradiction: suppose there is a point $z \in (-\frac{1}{2}\alpha m(I), \frac{1}{2}\alpha m(I))$ which is not in $F - F$. What does this say about the relation between F and its translate $F_z = \{x + z : x \in F\}$?

(II) It is enough to prove it for $E \subset [0, N)$ for all N (why?) If E is an interval $[a, b) \subset [0, N)$, how does squaring affect the measure? Use outer measure–consider the infimum over covers of a general null set E by sets of the form $[a, b)$.

(III)(a) It's enough to consider the case where the union is an interval (why?). One approach is to consider compact subsets of this interval.

(IV) Induction.