MATH 525a ASSIGNMENT 2 SOLUTIONS
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(5) $\mathcal{M} = \sigma(\mathcal{E})$. Let $\mathcal{G} = \{E \in \mathcal{M} : E \in \sigma(\mathcal{F})$ for some countable $\mathcal{F} \subset \mathcal{E}\}$. If $E \in \mathcal{G}$ then $E \in \sigma(\mathcal{F}) \subset \sigma(\mathcal{E}) = \mathcal{M}$ for some $\mathcal{F} \subset \mathcal{E}$. Thus $\mathcal{G} \subset \mathcal{M}$. Let us show $\mathcal{G}$ is a σ-algebra.

If $E \in \mathcal{G}$ then $\mathcal{E} \in \sigma(\mathcal{F})$ for some countable $\mathcal{F} \subset \mathcal{E}$, so $E^c \in \sigma(\mathcal{F})$, so $E^c \in \mathcal{G}$. Thus $\mathcal{G}$ is closed under complements.

If $E_1, E_2, ... \in \mathcal{G}$ then each $E_n \in \sigma(\mathcal{F}_N)$ for some countable $\mathcal{F}_n \subset \mathcal{E}$. Then $\cup_m \mathcal{F}_m$ is countable and each $E_n \in \sigma(\cup_m \mathcal{F}_m)$, so $\cup_n E_n \in \sigma(\cup_m \mathcal{F}_m)$, meaning $\cup_n E_n \in \mathcal{G}$. Thus $\mathcal{G}$ is closed under countable unions. It follows that $\mathcal{G}$ is a σ-algebra.

$\mathcal{G}$ contains $\mathcal{E}$, so $\mathcal{G} \supset \sigma(E) = \mathcal{M}$, so $\mathcal{G} = \mathcal{M}$.

(6) *Claim 1*: $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$.

Proof: Suppose $N \in \overline{\mathcal{M}}, A \subset N$ and $\overline{\mu}(N) = 0$. This means $N = E \cup F$ with $E \in \mathcal{M}$ and $F \subset N'$ for some $N' \in \mathcal{M}$ with $\overline{\mu}(N') = 0$. Then $\overline{\mu}(E) \leq \overline{\mu}(N) = 0$ so E is null, so $E \cup N'$ is null. Thus we can represent A as $\phi \cup A$ with $\phi \in \mathcal{M}$ and A contained in the null set $E \cup N' \in \mathcal{M}$. This says $A \in \overline{\mathcal{M}}$, so $\overline{\mu}$ is complete.

Claim 2: $\overline{\mu}$ is the only measure on $\overline{\mathcal{M}}$ that extends μ.

Proof: Suppose μ' is a measure on $\overline{\mathcal{M}}$ that extends μ. Let $E \cup F \in \overline{\mathcal{M}}$ with $E \in \mathcal{M}, F \subset N$ where $N \in \mathcal{M}$ is null. Then $F = \phi \cup F \in \overline{\mathcal{M}}$, and

$$\mu(E) = \mu'(E) \leq \mu'(E \cup F) \leq \mu'(E) + \mu'(F) \leq \mu'(E) + \mu'(N) = \mu(E) + \mu(N) = \mu(E),$$

so all of these are equal, meaning $\mu'(E \cup F) = \mu(E) = \overline{\mu}(E \cup F)$. Thus $\mu' = \overline{\mu}$, on $\mathcal{M}$.

(8) The sets $F_k = \cap_{n=k}^{\infty} E_n$ satisfy $F_1 \subset F_2 \subset ...$ so

$$\mu(\liminf_j E_j) = \mu(\cup_{k=1}^{\infty} F_k) = \lim_k \mu(F_k) \leq \liminf \mu(E_k).$$

The last inequality follows from $F_k \subset E_k$.

Similarly, the sets $G_k = \cup_{n=k}^{\infty} E_n$ satisfy $G_1 \supset G_2 \supset ...$with $\mu(G_1) < \infty$, so

$$\mu(\limsup_j E_j) = \mu(\cap_{k=1}^{\infty} G_k) = \lim_k \mu(G_k) \geq \limsup \mu(E_k).$$

The last inequality follows from $G_k \supset E_k$.

(9)

$$\mu(E) + \mu(F) = \mu(E \backslash F) + \mu(E \cap F) + \mu(F \backslash E) + \mu(E \cap F),$$

and the first three of the four measurable sets on the right side are disjoint with union $E \cup F$. Hence the right side is $\mu(E \cup F) + \mu(E \cap F)$.

(12)(a) $E \subset F \cup (E \triangle F)$, so $\mu(E) \le \mu(F) + \mu(E\triangle F) = \mu(F)$. Similarly $\mu(F) \le \mu(E)$, so $\mu(E) = \mu(F)$.

(b)(i) $\mu(E\triangle E) = \mu(\phi) = 0$ so $E \sim E$.

(ii) $E\triangle F = F\triangle E$ so $E \sim F$ if and only if $F \sim E$.

(iii) Suppose $\mu(E\triangle F) = \mu(F\triangle G) = 0$. Then

$$(E\backslash G) \cap F \subset F\backslash G \subset F\triangle G \quad \text{and} \quad (E\backslash G) \cap F^c \subset E\backslash F \subset E\triangle F.$$

Hence $E\backslash G \subset (F\triangle G) \cup (E\triangle F)$. Similarly, $G\backslash E \subset (F\triangle G) \cup (E\triangle F)$, so $E\triangle G \subset (F\triangle G) \cup (E\triangle F)$, and therefore $\mu(E\triangle G) \le \mu(F\triangle G) + \mu(E\triangle F) = 0$. Thus $E \sim G$.

(c) The last inequality in (b) says $\rho(E,G) \le \rho(E,F) + \rho(F,G)$.

(A) X must be a countable union of sets of finite measure, or equivalently, of finite sets. This means X must be countable.

(B) Let

$$E = \bigcup_{n\ge 1} \left[\frac{1}{2} + \frac{1}{2^{2n}}, \frac{1}{2} + \frac{1}{2^{2n-1}}\right).$$

Note E consists of a collection of intervals which "converge down toward 1/2," with gaps in between them. Then $\frac{1}{2} \in E^c$ but there is no interval $[\frac{1}{2}, \frac{1}{2} + \epsilon)$ contained in E^c. This means $E^c \notin \mathcal{M}$, so $\mathcal{M}$ is not a σ-algebra.

(C) Since $\mathcal{E} \subset \mathcal{F}$, we have $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$. Since $\sigma(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$, we have $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$. Therefore they are equal.

(D) We have $\mu(\phi) = \lim_n \mu_n(\phi) = 0$. Suppose $E_1, E_2, \dots$ are disjoint. Since $\mu_n(E_j) \le \mu(E_j)$, we have

$$\mu\left(\cup_{j=1}^\infty E_j\right) = \lim_n \mu_n\left(\cup_{j=1}^\infty E_j\right) = \lim_n \sum_{j=1}^\infty \mu_n(E_j) \le \sum_{j=1}^\infty \mu(E_j).$$

In the other direction, for each $k \ge 1$ we have

$$\mu\left(\cup_{j=1}^\infty E_j\right) = \lim_n \mu_n\left(\cup_{j=1}^\infty E_j\right) \ge \lim_n \mu_n\left(\cup_{j=1}^k E_j\right) = \lim_n \sum_{j=1}^k \mu_n(E_j) = \sum_{j=1}^k \mu(E_j).$$

Since k is arbitrary this means $\mu\left(\cup_{j=1}^\infty E_j\right) \ge \sum_{j=1}^\infty \mu(E_j)$, so we have equality, i.e. countable additivity holds.