MATH 525a ASSIGNMENT 1 SOLUTIONS
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(1)(a) Let

$$A = \{\text{all maps } \mathbb{N} \to \{0,1\}\} \quad \text{(sequences of 0's and 1's)}$$

$$B = \{\text{all maps } \mathbb{N} \to A\} \quad \text{(sequences of sequences)}$$

A map $f \in B$ corresponds to an infinite array with the sequence $f(n)$ as nth row. Then $\text{card}(A) = 2^{\aleph_0} = \mathfrak{c}$ and $\text{card}(B) = \mathfrak{c}^{\aleph_0}$. Define an injection $\varphi : B \to A$ as follows. Given an array (f_{ij}) corresponding to some $f \in B$, reorder it into a sequence by decomposing the array into lower-left-to-upper-right diagonals, then taking first the entries in the top diagonal, then the entries (from bottom to top) in the next diagonal, etc. The resulting sequence thereby defined to be $\varphi(f) \in A$. Clearly φ is a bijection, so $\text{card}(A) = \text{card}(B)$.

(b) The easiest injection may be as follows. Given $f = (n_1, n_2, ...) \in \mathbb{N}^{\mathbb{N}}$, write each n_i in binary and make a sequence $\psi(f) \in \{0,1,2\}^{\mathbb{N}}$ by stringing together the binary representations of the n_i's, using 2's to separate each n_i from the adjacent ones. For example if $f = (7, 17, 4, ...)$ then we convert to binary: $7 = 111$, $17 = 10001$, $4 = 100$, etc. and then $\psi(f) = (1, 1, 1, 2, 1, 0, 0, 0, 1, 2, 1, 0, 0, 2, ...)$.

(2) If $x_0 \in Y$ and $Y = g(x_0)$, then $x_0 \notin g(x_0) = Y$, a contradiction. If $x_0 \notin Y$ and $Y = g(x_0)$, then $x_0 \in g(x_0) = Y$, another contradiction. Thus $Y \neq g(x_0)$ for all x_0, so g is not onto.

(3) One example is $\{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 > 1\} \cup \{(1/2, 1/2)\}$. $z = (1/2, 1/2)$ is the unique minimal element but there is no x with $x \leq y$ for all $y \in X$.

(4)(a) Suppose f, g are both successor-preserving. If $f \neq g$, let w be the least element with $f(w) \neq g(w)$. Then $\{f(z) : z < w\} = \{g(z) : z < w\}$ so $f(w), g(w)$ are both the least element not in this set, meaning $f(w) = g(w)$, a contradiction. Thus $f = g$.

(b) Suppose $f(X) \subset Y, f(X) \neq Y$ and let u be the least element of Y not in $f(X)$. This means $v < u$ implies $v \in f(X)$. Conversely if $v \in f(X)$ then every $t < v$ is in $f(X)$, by the successor-preserving property. Since $u \notin f(X)$, we cannot have $u < v$ or $u = v$, so $u > v$. Thus $v \in f(X)$ if and only if $v < u$. Thus $f(X)$ is a segment, whenever $f(X)$ is not all of Y.

(5) Let $\{a_n\}, \{b_n\}$ be two sequences. Let $X = Y$ be the concatenation of the first two sequences, meaning the elements are in the order $a_1, a_2, \ldots b_1, b_2, \ldots$. Define $f : X \to Y$ by $f(a_n) = a_n, f(b_n) = b_{n+1}$. Then f is weakly successor-preserving because $f(b_{n+1}) = b_{n+2}$ is always the successor of $f(b_n) = b_{n+1}$. But f is not successor-preserving because the least element not in $\{f(z) : z < b_1\}$ is b_1, whereas $f(b_1) = b_2$ which is not the same.

A useful observation—this example works because $b_1 \in X$ is not the successor of any element of X.

(6) We can make an injection from X to Z by composing injections $f : X \to Y$ and $g : Y \to Z$, so we have $\text{card}(X) \leq \text{card}(Z)$. We need to show we do not have equality, that is, there is no injection from Z to X. Proceed by contradiction: suppose $h : Z \to X$ is an injection. Then $h \circ g$ is an injection from Y to X, so by Schroeder-Bernstein, $\text{card}(X) = \text{card}(Y)$, a contradiction.

(7) We may assume X, Y are disjoint (otherwise just replace Y with $Y \backslash X$.) Since X is infinite, there is a countably infinite $Z \subset X$. Since $Z \cup Y$ and Z are both countably infinite, there is a bijection $f : Z \cup Y \to Z$. We can extend this f to $X \cup Y$ by defining $f(x) = x$ for all $x \in X \backslash Z$. This extended f is a bijection from $X \cup Y$ to X.