MATH 525a ASSIGNMENT 1
FALL 2016
Prof. Alexander

Due Friday September 2. Most of these have fairly short answers!

Notation: A^B denotes the set of all mappings from B into A. If A has cardinality j and B has cardinality k then this is sometimes written as j^B or j^k. For example $2^{\mathbb{N}} = \{$all maps from $\mathbb{N}$ into $\{0,1\}\}$, or equivalently, all sequences of 0's and 1's.

(1) Let $\aleph_0$ (Hebrew letter aleph) denote the cardinality of $\mathbb{N} = \{1, 2, ...\}$ and let c denote the cardinality of $\mathbb{R}$.

(a) Show that $2^{\aleph_0} = c^{\aleph_0}$. That is, $\{$all sequences of 0's and 1's$\}$ and $\{$all sequences of reals$\}$ have the same cardinality. HINT: the set of all binary sequences (i.e. sequences of 0's and 1's) has cardinality c. (Why?)

(b) Since $2 < 3 < \aleph_0 < c$, it follows from (a) that $3^{\aleph_0} = (\aleph_0)^{\aleph_0}$. Construct an explicit injection of $\mathbb{N}^{\mathbb{N}}$ into $3^{\mathbb{N}}$. HINT: Use binary representations of elements of $\mathbb{N}$.

(2) In the proof of (0.9) in the text (page 7), there is a statement, "any attempt to answer the question 'is $x_0 \in Y$?' quickly leads to an absurdity." Fill in the details of this statement.

(3) Find a poset X which has a unique minimal element, but no element x for which $x \le y$ for every $y \in X$. HINT: Try a subset of the unit square.

(4) Let X be a well-ordered set. A function f from X to a well-ordered set Y is called *successor-preserving* if for all x, $f(x)$ is the least element of Y not in $\{f(z) : z < x\}$. A *segment* in Y is a set of form $\{u \in Y : u < y\}$ for some $y \in Y$.

(a) Show there is at most one successor-preserving mapping for any choice of X, Y.

(b) Show that the range of f is a segment, or all of Y.

(5) Let X and Y be well-ordered sets with minimal elements x_0 and y_0 respectively. The *successor* of an element $x \in X$ is the least element not in $\{z : z \le x\}$ (which exists, by well-ordering.) Let us call a function $f : X \to Y$ *weakly successor-preserving* if $f(x_0) = y_0$ and for every x with successor x', the successor of $f(x)$ is $f(x')$. Find an example of a weakly successor-preserving f which is not successor-preserving. HINT: One kind of well-ordered set is as follows: take two sequences $\{a_n\}$ and $\{b_n\}$ and concatenate them, that is, put them in the order $a_1, a_2, \dots b_1, b_2, \dots$, meaning $a_i \le b_j$ for all i, j. You can do similarly with more than 2 sequences.

(6) Suppose X, Y and Z are sets with $\text{card}(X) < \text{card}(Y)$ and $\text{card}(Y) \le \text{card}(Z)$. Show that $\text{card}(X) < \text{card}(Z)$.

(7) Suppose X is infinite and Y is at most countable. Show there exists a bijection between $X \cup Y$ and X. HINT: For a certain type of infinite X, this is straightforward. Use this to handle the general case. Note we don't assume X, Y disjoint.