MATH 525a MIDTERM SOLUTIONS
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(1)(a) In the text

(b) X is well-ordered if X is linearly ordered and every nonempty subset has a minimal element.

(2)(a) Let $\epsilon > 0$. Uniform continuity means there exists $\delta > 0$ such that $|y - x| < \delta \implies |F(y) - F(x)| < \epsilon$. Then since $f_n \to f$ in measure,

$$\mu\Big(\{x : |F(f_n(x)) - F(f(x))| \geq \epsilon\}\Big) \leq \mu\Big(\{x : |f_n(x) - f(x)| \geq \delta\}\Big) \to 0 \quad \text{as } n \to \infty.$$

This shows $F \circ f_n \to F \circ f$ in measure.

(b) For $\epsilon \in (0,1)$ we have $|\chi_{E_n}(x) - 1| > \epsilon \iff x \in E_n^c$, and therefore $\mu\big(\{x : |\chi_{E_n}(x) - 1| > \epsilon\}\big) = \mu\big(E_n^c\big)$. Hence $\chi_{E_n} \to 1$ in measure if and only if $\mu(E_n^c) \to 0$.

(c) From part (b), it's enough to show $\int_{E_n^c \cap D} g\ d\mu = \int_{E_n^c} g\ d\mu = \nu(E_n^c) \to 0$. Given $\epsilon > 0$, from homework, there exists $\delta > 0$ such that $\mu(A) < \delta \implies \int_A g\ d\mu < \epsilon$. By assumption there exists N such that $n \geq N \implies \mu(E_N^c \cap D) < \delta \implies \int_{E_n^c \cap D} g\ d\mu < \epsilon$. Since ϵ is arbitrary this shows $\int_{E_n^c \cap D} g\ d\mu \to 0$.

(3) Let $\mathcal{A}_{nk} = \{A \in \mathcal{A} : \mu(A \cap B_n) > 1/k\}$. Then

$$A \in \mathcal{A} \implies \mu(A) > \frac{1}{k} \text{ for some } k \implies \lim_n \mu(A \cap B_n) > \frac{1}{k} \text{ for some } k$$

$$\implies \mu(A \cap B_n) > \frac{1}{k} \text{ for some } n, k,$$

so $\mathcal{A} = \cup_{n,k} \mathcal{A}_{nk}$. The sets in $\mathcal{A}_{nk}$ are disjoint, so

$$\frac{1}{k}|\mathcal{A}_{nk}| \leq \sum_{A \in \mathcal{A}_{nk}} \mu(A \cap B_n) \leq \mu(B_n),$$

which shows that $\mathcal{A}_{nk}$ is finite. Thus $\mathcal{A}$ is a countable union of finite sets so is at most countable.

(4) It is enough to consider $f \geq 0$, since for general f we can apply the result to $f = f^+ - f^-$. Let $\epsilon > 0$. Since f is bounded, there exists a simple φ with $\sup_x |f(x) - \varphi(x)| < \epsilon$, say $\varphi = \sum_{k=1}^K c_k \chi_{E_k}$. Then

$$\int \varphi\ d\mu_n = \sum_{k=1}^K c_k \mu_n(E_k) \to \sum_{k=1}^K c_k \mu(E_k) = \int \varphi\ d\mu \quad \text{as } n \to \infty$$

and

$$\left|\int \varphi \ d\mu_n - \int f \ d\mu_n\right| \le \int |\varphi - f| \ d\mu_n \le \epsilon \mu_n(X),$$

$$\left|\int \varphi \ d\mu - \int f \ d\mu\right| \le \int |\varphi - f| \ d\mu \le \epsilon \mu(X),$$

so for large n,

$$\begin{aligned}\left|\int f \ d\mu_n - \int f \ d\mu\right| &\le \left|\int f \ d\mu_n - \int \varphi \ d\mu_n\right| + \left|\int \varphi \ d\mu_n - \int \varphi \ d\mu\right| + \left|\int \varphi \ d\mu - \int f \ d\mu\right| \\ &\le \epsilon \mu_n(X) + \epsilon + \epsilon \mu(X).\end{aligned}$$

Since $\mu_n(X) \to \mu(X)$, this shows $\limsup_n |\int f \ d\mu_n - \int f \ d\mu| \le \epsilon(2\mu(X) + 1)$. Since ϵ is arbitrary, this lim sup is 0, which shows $\int f \ d\mu_n \to \int f \ d\mu$.