MATH 525a MIDTERM EXAM
October 19, 2016
Prof. Alexander

Last Name: ______________________________

First Name: ______________________________

USC ID: ______________________________

Signature: ______________________________

Problem	Points	Score
1	22	
2	30	
3	23	
4	25	
Total	100	

Notes:

(1) This is a closed book exam–no books or notes allowed.
(2) Write on the backs of the sheets if you need more space. Do not use your own scratch paper.
(3) Cross out anything you don't want counted when the exam is graded.
(4) Longer problems, or parts of problems, have a $*$ by the problem number.

(1)(22 points)(a) Prove the following part of Proposition 1.10: Let $\mathcal{E} \subset \mathcal{P}(X)$ and let $\rho : \mathcal{E} \to [0, \infty]$ satisfy $\emptyset \in \mathcal{E}, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For all $A \subset X$ define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(E_j) : E_j \in \mathcal{E}, A \subset \cup_{j=1}^{\infty} E_j \right\}.$$

Show that μ^* is countably subadditive.

(b) For a set X ordered by a relation $\leq$, state what it means for X to be *well-ordered*.

(2)(30 points) Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Suppose f_n, f are real-valued measurable functions on X, $f_n \to f$ in measure, and $F : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Show that $F \circ f_n \to F \circ f$ in measure.

(b) Let $E_n, n \geq 1$ be measurable sets. What condition on the values $\mu(E_n)$ or $\mu(E_n^c)$ is equivalent to $\chi_{E_n} \to 1$ in measure? HINT: This isn't really related to part (a). Also, 1 means the constant function everywhere equal to 1.

(c*) Let $g \in L^1(\mu)$ be nonnegative, let $D = \{x : g(x) > 0\}$, and define $\nu(E) = \int_E g \, d\mu$. Let $E_n, n \geq 1$ be measurable sets and suppose $\mu(E_n^c \cap D) \to 0$. Show that $\chi_{E_n} \to 1$ in ν-measure. HINT: Note the convergence is in ν-measure, not μ-measure. You may use the fact from homework that given $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta \implies \int_A g \, d\mu < \epsilon$.

(3)(23 points) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $\mathcal{A}$ be a disjoint collection of measurable sets, each of strictly positive measure. Show that $\mathcal{A}$ is at most countable.

HINT: Let $B_1 \subset B_2 \subset \dots$ with $\mu(B_n) < \infty$ and $\cup_n B_n = X$. For given n, k consider the collection of sets $\{A \in \mathcal{A} : \mu(A \cap B_n) > 1/k\}$.

(4*)(25 points) Suppose μ_n, μ are finite measures on $(X, \mathcal{M})$, and $\mu_n(E) \to \mu(E)$ for all $E \in \mathcal{M}$. Show that for every bounded measurable $f : X \to \mathbb{R}$, we have $\int f \, d\mu_n \to \int f \, d\mu$.

HINT: You can approximate f by a simple function. Is the approximation uniform?