MATH 425a SAMPLE MIDTERM EXAM 2 SOLUTIONS
Fall 2016
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(1)(a) $\sum_n a_n$ converges if the sequence of partial sums $s_n = \sum_{i=1}^n a_i$ converges to a finite limit.

(b) See text.

(2)(a) Yes. If x is irrational then there is a sequence of rationals $x_n \to x$. Since f is continuous we have $f(x_n) \to f(x)$. But $f(x_n) = 0$ for all n so $f(x) = 0$, and this is valid for all irrational x.

(b) No. The Ratio Test establishes absolute convergence, so all rearrangements have limit s.

(3)(a) Compare to (constant)$/n^2$: there exists N such that

$$n \geq N \implies 2 < \frac{n^3}{2} \implies n^3 - 2 > \frac{n^3}{2} \implies \frac{n}{n^3-2} < \frac{n}{n^3/2} = \frac{2}{n^2}.$$

Since $\sum_n 2/n^2$ converges, it follows from the Comparison Test that $\sum_n n/(n^3-2)$ converges.

(b) Cauchy condensation test: terms are nonnegative and decreasing, and

$$\sum_k 2^k 2^{-\sqrt{\log_2 2^k}} = \sum_k 2^k 2^{-\sqrt{k}}.$$

This series diverges since terms do not $\to 0$. Therefore the original series diverges.

(c) Diverges since terms $\not\to 0$.

(d)

$$\left(n^2 \left(\frac{2}{n} \right)^n \right)^{1/n} = n^{2/n} \cdot \frac{2}{n} \to 1 \cdot 0 = 0,$$

so the radius of convergence is ∞.

(e) There exists N such that

$$n \geq N \implies \frac{a_n}{n} < 1 \implies \frac{1}{a_n} > \frac{1}{n}.$$

Since $\sum_n 1/n$ diverges, the Comparison Test says $\sum_n 1/a_n$ diverges.

(4)(a) Since f is uniformly continuous, given $\epsilon > 0$ there exists $\delta > 0$ such that $|p - p'| < \delta \implies |g(p) - g(p')| < \epsilon$. Since $\{p_n\}$ is Cauchy, there exists N such that

$$m, n \geq N \implies |p_m - p_n| < \delta \implies |g(p_m) - g(p_n)| < \epsilon.$$

This shows $\{g(p_n)\}$ is Cauchy.

(b) Let $\epsilon > 0$ and let δ be as in part (a). Then there exists N_1, N_2 such that

$$n \geq N_1 \implies |p_n| < \frac{\delta}{2} \quad \text{and} \quad n \geq N_2 \implies |t_n| < \frac{\delta}{2},$$

so

$$n \geq \max(N_1, N_2) \implies |p_n - t_n| < |p_n| + |t_n| < \delta \implies |g(p_n) - g(t_n)| < \epsilon.$$

(c) Since Cauchy sequences in $\mathbb{R}$ are convergent, by part (a) there exists q such that $g(p_n) \to q$. If any other sequence $t_n \to 0$ then by part (b),

$$|g(t_n) - q| \leq |g(t_n) - g(p_n)| + |g(p_n) - q| \to 0,$$

so $g(t_n) \to q$ also. By the hint, this means $\lim_{x \to 0} g(x) = q$.