MATH 425a ASSIGNMENT 9
FALL 2016 Prof. Alexander
Due Friday November 18.

Rudin Chapter 5 #1, 7, 13ab plus the problems (A)–(G) below:

(A) A function f on $\mathbb{R}$ is called *even* if $f(x) = f(-x)$ for all x. Show that if f is even, and differentiable at $x = 0$, then $f'(0) = 0$.

(B) Suppose f, g are differentiable on $[a, b]$, $f(a) = g(a)$, and $f' \leq g'$ on $[a, b]$. Show that $f(x) \leq g(x)$ for all $x \in [a, b]$.

(C) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable at x, and let $g(t) = f(t)^3$. Show *directly from the definition of derivative* (i.e. *not* using the product rule, chain rule, etc.) that $g'(x) = 3f(x)^2 f'(x)$.

(D) Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathbb{R}$ with $f''(x) > 0$ for all x. Show that for each $c \in \mathbb{R}$ there are at most two points where $f(x) = c$.

(E) Suppose f is differentiable in (a, b). State hypotheses under which $|f|$ is differentiable in (a, b), and prove it. Your hypotheses should not prohibit $f(x)$ from ever being 0.

(F) In class we will prove the following corollary to Theorem 5.8: if $f : [a, b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$, then there exists $x \in (a, b)$ with $f'(x) = 0$. Prove the following analog for a possibly infinite interval: suppose $f : (a, b) \to \mathbb{R}$ is differentiable, with $-\infty \leq a \leq b \leq \infty$, and suppose $\lim_{x\to a} f(x) = \lim_{x\to b} f(x) = L$ (a finite value.) Show that there exists $x \in (a, b)$ with $f'(x) = 0$.

(G) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

Find $f'(x)$ for all $x \in \mathbb{R}$, and show that $f''(0)$ does not exist.

HINTS:

(2) Use 4.17, and use the idea of substitution in limits.

(7) The derivatives are only assumed to exist at one point, so you can't use L'Hospital. Instead, what quantity do you know (essentially by definition) converges to $f'(x)/g'(x)$? Relate $f(t)/g(t)$ to this quantity.

(13b) Justify carefully—at $x = 0$ you can't just differentiate the formula, calculus-style.

(A) Use the definition of derivative.

(B) Apply the Mean Value Theorem (but not on the whole interval $[a, b]$) to one of the following functions: $h = f + g, h = f - g$ or $h = fg$.

(C) You can factor the difference of two cubes.

(D) Suppose there are three points, and get a contradiction. A picture may be helpful to let you see what is going on here, but a picture is not a proof.

(E) How is $\frac{d}{dx}|f(x)|$ related to $\pm f'(x)$, at points where $f(x) > 0$, and where $f(x) < 0$?

(F) Adapt the proof of the corollary, which is based on f having a local maximum or minimum.