MATH 425a ASSIGNMENT 8 SOLUTIONS
FALL 2015 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 4:

(18) We claim that $\lim_{x\to y} f(x) = 0$ for all $y \in \mathbb{R}$. To prove this, fix $y \in \mathbb{R}$ and $\epsilon > 0$ and let $N > 1/\epsilon$. Let $E = \{\frac{p}{q} \in \mathbb{Q} : 0 < |\frac{p}{q} - y| < 1, 1 \le q \le N\}$. (That is, E contains rationals with denominator at most N.) Then E is finite and $y \notin E$, so the closest point to y in E is at a positive distance δ from y. This means that for x with $0 < |x - y| < \delta$, we either have $f(x) = 0$ (if x is irrational) or $f(x) = 1/n$ with $n > N$ (if x is rational.) Thus we have

$$0 < |x - y| < \delta \implies |f(x)| < \frac{1}{N} < \epsilon.$$

This shows that indeed $\lim_{x\to y} f(x) = 0$.

It follows that f is continuous at $y \iff f(y) = 0 \iff y$ is irrational, and otherwise the discontinuity is simple, since $\lim_{x\to y} f(x)$ exists.

Handout:

(A) Let $\epsilon > 0$. Then there exist δ_1, δ_2 such that

$$x, y \in [a, c], |x-y| < \delta_1 \implies |f(x)-f(y)| < \epsilon, \quad x, y \in [c, b], |x-y| < \delta_2 \implies |f(x)-f(y)| < \epsilon.$$

If $x \in [a, c], y \in [c, b]$ with $|x - y| < \min(\delta_1, \delta_2)$, then $|x - c| < \delta_1$ and $|y - c| < \delta_2$ so

$$|f(x) - f(y)| \le |f(x) - f(c)| + |f(c) - f(y)| < \epsilon + \epsilon = 2\epsilon.$$

Thus $\delta = \min(\delta_1, \delta_2)$ "works" (for 2ϵ in place of ϵ) throughout $[a, b]$, so f is uniformly continuous there.

(B)(a) f, g bounded means there exists M such that $|f(x)| \le M$ and $|g(x)| \le M$, for all x. Let $\epsilon > 0$. f, g uniformly continuous means there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon \quad \text{and} \quad |g(x) - g(y)| < \epsilon.$$

Then for $|x - y| < \delta$,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\le |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &= |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\ &\le M\epsilon + M\epsilon \\ &= 2M\epsilon. \end{aligned}$$

This shows that δ "works" for $2M\epsilon$ in place of ϵ. Since ϵ is arbitrarily small, so is $2M\epsilon$, so this proves uniform continuity of fg.

(b) One example: let $f(x) = g(x) = x$ on $[0, \infty)$. Then given $\epsilon > 0$, we have $|x - y| < \epsilon \implies |f(x) - f(y)| < \epsilon$, so $\delta = \epsilon$ "works", so f (and also g) is uniformly continuous. To show $(fg)(x) = x^2$ is not uniformly continuous, take $\epsilon = 1$ and let $\delta > 0, x > 1/\delta$ and $y = x + \frac{\delta}{2}$. Then $|y - x| < \delta$ but

$$|(fg)(y) - (fg)(x)| = |y^2 - x^2| = x\delta + \frac{\delta^2}{4} > x\delta > 1 = \epsilon.$$

Thus δ does not "work" for all x, y. Since δ is arbitrary, this shows fg is not uniformly continuous.

(C)(a) Let $\epsilon > 0$. Since $f(x) \to L$ as $x \to \infty$, there exists M such that $x \geq M \implies |f(x) - L| < \epsilon/2$. Therefore

$$x, y \geq M \implies |f(x) - f(y)| \leq |f(x) - L| + |f(y) - L| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Since f is continuous and $[0, M+1]$ is compact, f is uniformly continuous on $[0, M+1]$, so there exists $\delta > 0$ such that

$$x, y \in [0, M+1], |y - x| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since any smaller δ also "works," we may assume $\delta < 1$. Then whenever $x, y \in [0, \infty)$ with $|x - y| < \delta$, we either have both $x, y \in [0, M+1]$ or both $x, y \in [M, \infty)$, so either way, we have $|f(x) - f(y)| < \epsilon$. This shows f is uniformly continuous.

(b) f is continuous, and $f(x) \to 0$ as $x \to \infty$, so f is uniformly continuous by (a).

(D) Since f is continuous, so is $|f|$, by exercise (V) in Assignment 7. Let $\epsilon = \inf_{x \in K} |f(x)|$. Since K is compact, there exists $x \in K$ with $|f(x)| = \epsilon$. Thus either $\epsilon = 0$, in which case $f(x) = 0$, or $\epsilon > 0$, in which case $|f(p)| \geq \epsilon > 0$ for all $p \in K$, meaning f is bounded away from 0.

(E)(a) One example: For $f(x) = 1/(1 + x^2)$, we have $f^{-1}([0, 1]) = \mathbb{R}$ which is not compact.

(b) One example: For $f(x) = e^x$, $(-\infty, 0]$ is closed but $f((-\infty, 0]) = (0, 1]$ is not.

(F) This is false. For example, let $X = [0, 2\pi)$ and $Y = \{z \in \mathbb{C} : |z| = 1\}$, and define $f : X \to Y$ by $f(x) = e^{ix}$. Then f is continuous and a bijection, but if we take $\{x_n\}$ with the odd terms $x_{2n+1} \nearrow 2\pi$ and the even terms $x_{2n} \searrow 0$, then the full sequence $f(x_n) \to 1$, but the inverse images $\{x_n\}$ do not converge. (This same f was used in lecture to show the inverse of a continuous bijection does not have to be continuous.)

(G)(a) Every point of $\mathbb{Z}$ is isolated, and every function is continuous at each isolated point in its domain, so all $f : \mathbb{Z} \to Y$ are continuous.

(b) Let $p \in X$ be a limit point of X. Define

$$f(x) = \begin{cases} 1 & \text{if } x \neq p, \\ 0 & \text{if } x = p. \end{cases}$$

Then $\lim_{x \to p} f(x) = 1$ but $f(p) = 0$, so f is not continuous at p.