MATH 425a ASSIGNMENT 7 SOLUTIONS
FALL 2016 Prof. Alexander

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Rudin Chapter 3 and 4:

(11)(c) We have

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{a_n}{s_n s_{n-1}} \geq \frac{a_n}{s_n^2},$$

and

$$\sum_{n=2}^{N} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = \frac{1}{s_1} - \frac{1}{s_N} \to \frac{1}{s_1} \quad \text{as } N \to \infty,$$

meaning

$$\sum_{n=2}^{\infty} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right)$$

converges. Therefore $\sum \frac{a_n}{s_n^2}$ converges, by the Comparison Test.

(2) Let $f : X \to Y$ be continuous, and $E \subset X$. We have $f(\overline{E}) = f(E) \cup f(E')$, and certainly $f(E) \subset \overline{f(E)}$, so we need to show $f(E') \subset \overline{f(E)}$.

Let $p \in E'$; then there exists $p_n \to p$ with $p_n \in E$. From 4.2 we have $f(p_n) \to f(p)$, so either $f(p)$ is a point of $f(E)$ or $f(p)$ is a limit point of $f(E)$. Either way we have $f(p) \in \overline{f(E)}$, and this is true for all $p \in E'$, so we have $f(E') \subset \overline{f(E)}$ as desired.

(4) Let $f : X \to Y$ be continuous, and let E be dense in X. We want to show that if $y \in f(X)$ then either $y \in f(E)$ or $y \in f(E)'$. Suppose $y \in f(X)$ but $y \notin f(E)$. Then $y = f(x)$ for some $x \in E^c$. Since E is dense in X, x must be a limit point of E, so there is a sequence $\{x_n\}$ in E with $x_n \to x$. Then since f is continuous, we have $f(x_n) \to f(x) = y$. Since $f(x_n) \in f(E)$ and $y \notin f(E)$, we must have $f(x_n) \neq y$, so every neighborhood of y contains points $f(x_n) \neq y$ with $f(x_n) \in f(E)$, which shows that $y \in f(E)'$.

Thus $y \in f(X)$ implies that either $y \in f(E)$ or $y \in f(E)'$, meaning that $f(E)$ is dense in $f(X)$.

Handout:

(I) Since $\sum a_n$ converges, its partial sums form a bounded sequence. Therefore we can apply 3.42 with $b_n = 1/n$ and obtain that $\sum \frac{a_n}{n} = \sum a_n b_n$ converges.

(II) Given $\epsilon > 0$, let $\delta = \min(\frac{1}{2}, \frac{\epsilon}{2})$. Then $|x| < \delta \implies |1 - x| > 1/2$, so

$$|x| < \delta \implies \left| \frac{x}{1-x} \right| < \frac{\frac{\epsilon}{2}}{\frac{1}{2}} = \epsilon.$$

This shows $\lim_{x\to 0} \frac{x}{1-x} = 0$.

(III) It is enough to show that for every p, $\lim_{x\to p} f(x)$ does not exist. Suppose this limit existed for some p, with value L. Let $\epsilon < \frac{1}{2}$. Then there exists $\delta > 0$ such that $x \in (p-\delta, p+\delta) \implies |f(x) - L| < \epsilon < \frac{1}{2}$. There exist rational values in $(p-\delta, p+\delta)$ where $f(x) = 0$, and irrational values where $f(x) = 1$, so both $|0 - L| < \frac{1}{2}$ and $|1 - L| < \frac{1}{2}$. There is no such value L, so we have a contradiction, meaning the limit does not exist.

(IV)(a) True. The series converges, by the Alternating Series Test. It does not converge absolutely, since $\sum_n n^{-1} = \infty$. Therefore it converges conditionally. By Theorem 3.54 it has a rerrangement which converges to any given number, in particular to 3.14159.

(b) True. If a series is conditionally convergent, by Theorem 3.54 it has a rearrangement which diverges to $+\infty$. So if a series has no such divergent rearrangement, it must be absolutely convergent.

(c) False. Let $\sum_n b_n$ be any conditionally convergent series. Then by Theorem 3.54 it has a rearrangement $\sum_n a_n$ which diverges to $+\infty$. This means $\sum_n a_n$ is divergent but has a convergent rearrangement $\sum_n b_n$.

(V)(a) Suppose f is continuous. We need to show that $\lim_{x\to p} |f(x)| = |f(p)|$, for all p. From Chapter 1 #13 (Assignment 2), we have $\big||f(x)| - |f(p)|\big| \leq |f(x) - f(p)|$ so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - p| < \delta \implies |f(x) - f(p)| < \epsilon \implies \big||f(x)| - |f(p)|\big| < \epsilon.$$

This shows that $|f|$ is continuous at p.

(b) No. For example let

$$f(x) = \begin{cases} -1 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

As in problem (III), f is not continuous at any p. But $|f(x)| \equiv 1$ so $|f|$ is continuous.

(VI)(a) Let $f(x) = \cos(1/x)$, $x_n = 1/2n\pi$ and $t_n = 1/(2n+1)\pi$. Then $f(x_n) = 1$ for all n, so $f(x_n) \to 1$, while $f(t_n) \to -1$. Since these limits are unequal, it follows from Theorem 4.2 that $\lim_{x\to 0} f(x)$ does not exist.

(b) For $x \neq 0$ we have $|x \sin \frac{1}{x}| \leq |x|$. Therefore given $\epsilon > 0$, taking $\delta = \epsilon$ we have

$$0 < |x| < \epsilon \implies |x \sin \frac{1}{x}| \leq |x| < \epsilon.$$

This shows that $\lim_{x\to 0} x \sin(1/x) = 0$.

(VII)(a) Given $\epsilon > 0$, if $d(x,p) < \epsilon/5$ then $d(f(x), q) < \epsilon$. This shows that $\delta = \epsilon/5$ "works". Thus $\lim_{x\to p} f(x) = q$.

(b) Let $q > c$, and choose $\epsilon > 0$ satisfying $q - \epsilon > c$. If $\lim_{x\to p} f(x) = q$ then there exists $\delta > 0$ such that

$$x \in E, 0 < d(x,p) < \delta \implies |f(x) - q| < \epsilon \implies f(x) > q - \epsilon > c.$$

This is not possible since $f(x) \leq c$ for all x, so we cannot have $\lim_{x \to p} f(x) = q$. This is true for all $q > c$ so $\lim_{x \to p} f(x) \leq c$.