MATH 425a ASSIGNMENT 6 SOLUTIONS
FALL 2016 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 3:

(10) Suppose $a_n \in \mathbb{Z}$ and $a_n \neq 0$ for infinitely many n, say for indices $n_1 < n_2 < \dots$. Then $|a_{n_k}| \geq 1$ for all k, so $|a_{n_k}|^{1/n_k} \geq 1$. The existence of such a subsequence shows that $\limsup_{k\to\infty} |a_n|^{1/n} \geq 1$, so $R \leq 1$.

(21) Suppose X is complete and $E_1 \supset E_2 \supset \dots$, with E_n nonempty, closed and bounded, satisfying $\text{diam}(E_n) \to 0$. Let $x_n \in E_n$. Given $\epsilon > 0$, there exists N such that $\text{diam}(E_N) < \epsilon$. Then for $n, m \geq N$ we have $x_n, x_m \in E_N$ so $d(x_n, x_m) < \epsilon$. This shows that $\{x_n\}$ is Cauchy; since X is complete we have $x_n \to x$ for some x. For fixed k we have $x_k, x_{k+1}, \dots \in E_k$, $x_k \to x$ and E_k closed, so $x \in E_k$, for all k. Thus $x \in \cap_{k=1}^\infty E_k$. Now for all n, $\text{diam}(\cap_{k=1}^\infty E_k) \leq \text{diam}(E_n) \to 0$ as $n \to \infty$, so $\text{diam}(\cap_{k=1}^\infty E_k) = 0$. But this means $\cap_{k=1}^\infty E_k$ cannot contain more than one point, so this intersection is $\{x\}$.

Handout:

(A) Since $\sum a_n$ converges, we have $a_n \to 0$, so there exists N such that $n \geq N \implies a_n < 1 \implies a_n^2 < a_n$. Therefore $\sum a_n^2$ converges by the Comparison Test.

(B)(a) Suppose that for some $c > 0$ and $\gamma > 0$ we know that the inequality

$$(*) \qquad a_n \geq c\gamma^n.$$

holds for both n and $n+1$. We would like to show that it then holds for $n+2$. The validity of $(*)$ for n and $n+1$ implies that

$$a_{n+2} = a_{n+1} + a_n \geq c\gamma^{n+1} + c\gamma^n,$$

so to conclude that $(*)$ holds for $n+2$ it is sufficient that the right side of this inequality satisfy

$$c\gamma^{n+1} + c\gamma^n \geq c\gamma^{n+2}.$$

We can divide out $c\gamma^n$ so this is equivalent to $\gamma + 1 \geq \gamma^2$. The quadratic formula gives us the roots of $\gamma + 1 = \gamma^2$, and we conclude that $\gamma + 1 \geq \gamma^2$ is valid for all $1 < \gamma \leq (1+\sqrt{5})/2$.

For such γ, we have shown that if $(*)$ holds for n and $n+1$, it also holds for $n+2$. To prove by induction that $(*)$ holds for all n, we need to start off the induction by showing it holds for $n = 1, 2$. This means that we need to show

$$1 = a_1 \geq c\gamma \quad \text{and} \quad 1 = a_2 \geq c\gamma^2.$$

But these are both true provided we choose $c < \min(1/\gamma, 1/\gamma^2)$.

Thus for all $1 < \gamma \le (1+\sqrt{5})/2$ and $0 < c < \min(1/\gamma, 1/\gamma^2)$, $(*)$ holds for all n.

(b) By (a) we have $1/a_n \le 1/c\gamma^n$, and $\sum \frac{1}{c\gamma^n}$ converges (geometric series with $1/\gamma < 1$), so by the Comparison Test, $\sum \frac{1}{a_n}$ converges.

(c) From (a), taking $\gamma_0 = (1+\sqrt{5})/2$ we have

$$n^{-1}\log a_n \ge n^{-1}\log(c\gamma_0^n) = n^{-1}(\log c + n\log\gamma_0) = n^{-1}\log c + \log\gamma_0. \tag{1}$$

In the other direction, as in (a), consider the inequality

$$(**) \qquad a_n \le \gamma_0^n.$$

Since $a_1 = a_2 = 1$, this is true for $n = 1, 2$. Suppose $(**)$ is true for some integers $n, n+1$. Since γ_0 is a root of $\gamma^2 - \gamma - 1$, we have $\gamma_0 + 1 = \gamma_0^2$. Therefore

$$a_{n+2} = a_{n+1} + a_n \le \gamma_0^{n+1} + \gamma_0^n = \gamma_0^n(\gamma_0 + 1) = \gamma_0^{n+2},$$

so $(**)$ holds for $n+2$. Thus by induction, $(**)$ holds for all n, and therefore

$$n^{-1}\log c + \log\gamma_0 \le n^{-1}\log a_n \le n^{-1}\log(\gamma_0^n) = \log\gamma_0.$$

It follows that

$$\left|\log\gamma_0 - n^{-1}\log a_n\right| \le \left|n^{-1}\log c\right| \to 0 \quad \text{as } n \to \infty,$$

that is, $n^{-1}\log a_n \to \log\gamma_0$.

(C) Suppose first that L is finite. Let $\epsilon > 0$. There exists N_1 such that $n \ge N_1 \implies \left|\frac{|a_{n+1}|}{|a_n|} - L\right| < \epsilon$. Hence for $n \ge N_1$ we have

$$|a_n| = |a_{N_1}| \prod_{j=N_1}^{n-1} \frac{|a_{j+1}|}{|a_j|} \le |a_{N_1}|(L+\epsilon)^{n-N_1} \tag{2}$$

so $|a_n|^{1/n} \le |a_{N_1}|^{1/n}(L+\epsilon)^{1-N_1/n} \to L+\epsilon$ as $n \to \infty$. It follows that $\limsup |a_n|^{1/n} \le L+\epsilon$ and then since ϵ is arbitrary, $\limsup |a_n|^{1/n} \le L$.

If $L = 0$, this shows $\limsup |a_n|^{1/n} = L$ and we are done. If $L > 0$, we can repeat the above reasoning with $0 < \epsilon < L$ to obtain that for $n \ge N_1$,

$$|a_n| = |a_{N_1}| \prod_{j=N_1}^{n-1} \frac{|a_{j+1}|}{|a_j|} \ge |a_{N_1}|(L-\epsilon)^{n-N_1}. \tag{3}$$

As above it follows that $\limsup |a_n|^{1/n} \ge L-\epsilon$ and then since ϵ is arbitrary, $\limsup |a_n|^{1/n} \ge L$. Therefore $\limsup |a_n|^{1/n} = L$.

Next suppose $L = +\infty$. Then given $R > 0$ there exists N_2 such that $n \ge N_2 \implies \frac{|a_{n+1}|}{|a_n|} > R$, so as in (3), $|a_n| \ge |a_{N_1}|R^{n-N_1}$, and then $\limsup |a_n|^{1/n} \ge R$. Since R is arbitrary this shows $\limsup |a_n|^{1/n} = +\infty$.

(D)(a) $\lim_n \left(\frac{1}{n^2}\right)^{1/n} = \lim_n \frac{1}{n^{2/n}} = 1$ so $R = 1$.

(b) $\lim_n \left(\frac{1}{(\log n)^n}\right)^{1/n} = \lim_n \frac{1}{\log n} = 0$ so $R = \infty$.

(c) $\lim_n \left(\frac{n^4}{3^n}\right)^{1/n} = \lim_n \frac{n^{4/n}}{3} = \frac{1}{3}$ so $R = 3$.

(d) We use (C). We have $\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = (1+\frac{1}{n})^n \to e$ so $R = 1/e$.

(e) We use (C) again. We have $\frac{|a_{n+1}|}{|a_n|} = \frac{((n+1)!)^2}{(2n+2)!}\frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{(n+1)^2}{(2n+2)^2}\frac{2n+2}{2n+1} = \frac{1}{4}\frac{2n+2}{2n+1} \to \frac{1}{4}$, so $R = 4$.

(E)(a) Since $n-1 \geq n/2$ for all $n \geq 2$, we have $\frac{1}{n(n-1)} \leq \frac{2}{n^2}$. Since $\sum \frac{2}{n^2}$ converges, so does $\sum \frac{1}{n(n-1)}$ by the Comparison Test.

(b) Since $n^2+1 \leq 2n^2$ for all $n \geq 1$, we have $\sqrt{\frac{6n}{n^2+1}} \geq \sqrt{\frac{6n}{2n^2}} = \frac{\sqrt{3}}{\sqrt{n}}$. Since $\sum \frac{\sqrt{3}}{\sqrt{n}}$ diverges, so does $\sum \sqrt{\frac{6n}{n^2+1}}$, by the Comparison Test.

(c) Let $q \in (1,p)$. Since $\frac{\log n}{n^{p-q}} \to 0$, there exists N such that $n \geq N \implies \log n \leq n^{p-q} \implies \frac{\log n}{n^p} \leq \frac{1}{n^q}$. Since $q > 1$, the series $\sum \frac{1}{n^q}$ converges, hence so does $\sum \frac{\log n}{n^p}$ by the Comparison Test.

(F)(a) If $s_{2n} \to s$ then $s_{2n+1} = s_{2n} + a_{2n+1} \to s + 0 = s$ also. Hence given $\epsilon > 0$ there exist N_1, N_2 such that

$$n \geq N_1, n \text{ even} \implies |s_n - s| < \epsilon, \quad n \geq N_2, n \text{ odd} \implies |s_n - s| < \epsilon,$$

the full sequence $s_n \to s$, that is, the series converges to s. ADD MORE

(b) This is false. For example take $a_n = (-1)^n$. Then $s_{2n-1} = -1, s_{2n} = 0$ for all n, so both the subsequences are constant, hence convergent. But the full sequence $\{s_n\}$ does not converge.

(G) We first determine for which n the given term can be compared to a_n. In fact we have

$$\frac{a_n^{2/5}}{n^{2/3}} \leq a_n \iff a_n \geq \frac{1}{n^{10/9}}.$$

For other n (that is, those with $a_n < \frac{1}{n^{10/9}}$) we have

$$\frac{a_n^{2/5}}{n^{2/3}} \leq \frac{1}{n^{4/9}}\frac{1}{n^{2/3}} = \frac{1}{n^{10/9}}$$

Therefore

$$\frac{a_n^{2/5}}{n^{2/3}} \leq \begin{cases} a_n & \text{if } a_n \geq \frac{1}{n^{10/9}}, \\ \frac{1}{n^{10/9}} & \text{if } a_n < \frac{1}{n^{10/9}} \end{cases}$$
$$\leq a_n + \frac{1}{n^{10/9}} \text{ for all } n.$$

Since $\sum_n (a_n + n^{-10/9})$ converges, so does $\sum_n a_n^{2/5}/n^{2/3}$ by the Comparison Test.

(H)(a) First simplify each term algebraically:

$$\begin{aligned}
&\sum_{k=1}^{\infty} \left(\frac{a_1}{3k-2} + \frac{a_2}{3k-1} + \frac{a_3}{3k} - \frac{a_1 + a_2 + a_3}{3k} \right) \qquad (4) \\
&\quad = \sum_{k=1}^{\infty} \left(a_1 \left(\frac{1}{3k-2} - \frac{1}{3k} \right) + a_2 \left(\frac{1}{3k-1} - \frac{1}{3k} \right) \right) \\
&\quad = \sum_{k=1}^{\infty} \left(\frac{2a_1}{3k(3k-2)} + \frac{a_2}{3k(3k-1)} \right)
\end{aligned}$$

Now compare both terms to (constant)/k^2: for all $k \geq 1$ we have

$$3k - 2 \geq k \quad \text{and} \quad 3k - 1 \geq 2k,$$

and therefore

$$\left| \frac{2a_1}{3k(k-2)} + \frac{a_2}{3k(3k-1)} \right| \leq \frac{2|a_1|}{3k(k-2)} + \frac{|a_2|}{3k(3k-1)} \leq \frac{2|a_1|}{3k \cdot k} + \frac{|a_2|}{3k \cdot 2k} = \frac{4|a_1| + |a_2|}{6} \frac{1}{k^2}.$$

Since $\sum_k C/k^2$ converges for any constant C, the convergence of the original series (4) follows from the Comparison Test.

(b) If the series

$$\sum_{k=1}^{\infty} \left(\frac{a_1}{3k-2} + \frac{a_2}{3k-1} + \frac{a_3}{3k} \right) \qquad (5)$$

converged, we could subtract the convergent series (4) (in part (a)) from it, and as a result the series

$$\sum_{k=1}^{\infty} \frac{a_1 + a_2 + a_3}{3k}$$

would converge. But this is a nonzero multiple of the harmonic series, so it does not converge. Therefore the series (5) must not converge.

(c) When $a_1 + a_2 + a_3 = 0$, the convergent series (4) in part (a) is the same as (5), so (5) converges.