MATH 425a ASSIGNMENT 6
FALL 2016 Prof. Alexander
Due Friday October 21.

Rudin Chapter 3 #10, 21 plus the problems (A)–(G) below:

(A) Suppose $a_n > 0$ and $\sum_n a_n$ converges. Show that $\sum_n a_n^2$ converges.

(B) Let $\{a_n\}$ be the Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$ given by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$.

(a) Find numbers $c > 0$ and $1 < \gamma < 2$ such that $a_n \geq c\gamma^n$ for all $n \geq 1$.

(b) Show that $\sum_n \frac{1}{a_n}$ converges.

(c)(Harder bonus problem!) Show that $\lim_n n^{-1} \log a_n = \log \frac{1+\sqrt{5}}{2}$. HINT: Look carefully at what you did in (a).

(C) Let $\{a_n\}$ be a sequence in $\mathbb{R}$, and suppose $L = \lim_n |a_{n+1}/a_n|$ exists. Show that $\lim_n |a_n|^{1/n} = L$ also. (Note this gives an alternate way of finding the radius of convergence of a power series. It also shows that the root test is stronger than the ratio test.)

(D) Find the radius of convergence. Problem (C) may be useful for some.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ (b) $\sum_{n=2}^{\infty} \frac{x^n}{(\log n)^n}$ (c) $\sum_{n=0}^{\infty} \frac{n^4}{5^n} x^n$ (d) $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$ (e) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

(E) Establish convergence or divergence:

(a) $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ (b) $\sum_{n=1}^{\infty} \sqrt{\frac{6n}{n^2+1}}$ (c) $\sum_{n=1}^{\infty} \frac{\log n}{n^p}$, with $p > 1$.

(F) Let $\{a_n\}$ be a sequence and $s_n = \sum_{i=1}^{n} a_i$.

(a) If the sequence $s_2, s_4, s_6, ...$ of even partial sums converges, and $a_n \to 0$, show that $\sum a_n$ converges.

(b) Prove by logic, or disprove by example: without assuming $a_n \to 0$, if the subsequences $s_1, s_3, s_5, ...$ and $s_2, s_4, s_6, ...$ are both convergent, then $\sum a_n$ converges.

(G) Suppose $a_n \geq 0$ and $\sum_n a_n$ converges. Show that the series

$$\sum_n \frac{a_n^{2/5}}{n^{2/3}}$$

converges.

(H) Let $a_1, a_2, a_3 \in \mathbb{C}$.

(a) Show that

$$\sum_{k=1}^{\infty} \left(\frac{a_1}{3k-2} + \frac{a_2}{3k-1} + \frac{a_3}{3k} - \frac{a_1 + a_2 + a_3}{3k} \right)$$

converges.

(b) Suppose $a_1 + a_2 + a_3 \neq 0$. Show that

$$\sum_{k=1}^{\infty} \left(\frac{a_1}{3k-2} + \frac{a_2}{3k-1} + \frac{a_3}{3k} \right)$$

diverges. (Note that this is like the series $\sum_n 1/n$, but with each term multiplied successively by $a_1, a_2, a_3, a_1, a_2, a_3, \ldots$, and the terms then grouped into threes.)

(c) Suppose $a_1 + a_2 + a_3 = 0$. Show that

$$\sum_{k=1}^{\infty} \left(\frac{a_1}{3k-2} + \frac{a_2}{3k-1} + \frac{a_3}{3k} \right)$$

converges.

HINTS:

(21) This one is challenging. Make use of a sequence consisting of one point out of each E_n.

(B)(a) Suppose the relation $a_n \geq c\gamma^n$ holds for some indices n and $n+1$. Try to use this to obtain the same relation for index $n+2$, using $a_{n+2} = a_{n+1} + a_n$. You will find this only works if γ satisfies a certain inequality relating $1, \gamma$ and γ^2. Figure out what γ's satisfy this inequality, and pick any one such γ. Now use induction to establish the relation for all n.

(C) Fix $\epsilon > 0$ and show separately that $|a_n|^{1/n} < L + \epsilon$ and $|a_n|^{1/n} > L - \epsilon$, for large enough n.

(F) Consider the situation in which each subsequence is constant.

(G) Try the "Comparison Test with two bounds" method shown in lecture. When can you bound $a_n^{2/5}/n^{2/3}$ by a_n?

(H)(a) Simplify algebraically. (b) If this series converged, what other series would have to converge, because of part (a)? Does that series actually converge?