MATH 425a ASSIGNMENT 5 SOLUTIONS
FALL 2016 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 3:

(3) We claim that for all $n \geq 1$,

$$(*) \qquad s_n < 2 \quad \text{and} \quad s_n \leq s_{n+1}.$$

We check for $n = 1$: clearly $s_1 = \sqrt{2} < 2$, and $s_2 > \sqrt{2} = s_1$, so $(*)$ is true for $n = 1$. Suppose it is true for some n. Now

$$s_n \leq s_{n+1} \implies \sqrt{s_n} \leq \sqrt{s_{n+1}} \implies \sqrt{2+\sqrt{s_n}} \leq \sqrt{2+\sqrt{s_{n+1}}} \implies s_{n+1} \leq s_{n+2},$$

and similarly

$$s_n < 2 \implies \sqrt{s_n} < \sqrt{2} \implies s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+\sqrt{2}} < 2,$$

so $(*)$ is true for $n+1$. Thus by induction, $(*)$ is true for all $n \geq 1$. It follows that $\{s_n\}$ is a bounded monotone increasing sequence, so it must converge, by 3.14.

(5) Let $\alpha = \limsup a_n, \beta = \limsup b_n$.

Suppose first that neither α nor β is $+\infty$. Let $r > \alpha$ and $s > \beta$. From 3.17, there exist N_1, N_2 such that

$$n \geq N_1 \implies a_n < r, \qquad n \geq N_2 \implies b_n < s.$$

Then $n \geq \max(N_1, N_2) \implies a_n + b_n < r + s$, so there are only finitely many values $a_n + b_n \geq r+s$. This means $\{a_n + b_n\}$ has no subsequential limits above $r+s$, so $\limsup_n (a_n + b_n) \leq r + s$. Since $r > \alpha$ and $s > \beta$ are arbitrary, it follows that $\limsup_n (a_n + b_n) \leq \alpha + \beta$.

If one of α, β is $+\infty$ and the other is not $-\infty$, then the right side $\alpha + \beta$ of the desired inequality is $+\infty$ so there is nothing to prove.

Handout:

(I)(a) Let $\epsilon > 0$. There exist N_1 such that $n \geq N_1 \implies |s_n - 2| < \epsilon$, and K_1 such that $k \geq K_1 \implies |s_{n_k} + t_{n_k} - c| < \epsilon$, and K_3 such that $k \geq K_3 \implies n_k \geq N_1 \implies |s_{n_k} - 2| < \epsilon$. Let $K = \max(K_2, K_3)$. For $k \geq K$ we have

$$|t_{n_k} - (c-2)| = |s_{n_k} + t_{n_k} - c - (s_{n_k} - 2)| \leq |s_{n_k} + t_{n_k} - c| + |s_{n_k} - 2| < 2\epsilon.$$

Since ϵ is arbitrary this shows $t_{n_k} \to c - 2$.

(b) If c is a subsequential limit of $\{s_n + t_n\}$, then by (a), $c - 2$ is a subsequential limit of $\{t_n\}$, so $c - 2 \leq 3$, so $c \leq 5$. This shows that $\limsup_n (s_n + t_n) \leq 5$.

(II) Since $p \in G$ and G is open, there is a neighborhood $N_r(p) \subset G$. Since $p_n \to p$, there exists N such that $n \geq N \implies d(p_n, p) < r \implies p_n \in N_r(p) \implies p_n \in G$. Therefore at most $N - 1$ points p_n are not in G.

(III) There exists a subsequence $t_{n_k} \to \alpha$, and since $s_n \to s$ we have $s_{n_k} \to s$ as well. Therefore $s_{n_k} + t_{n_k} \to s + \alpha$, which shows that $\limsup(s_n + t_n) \geq s + \alpha$. The opposite inequality, $\limsup(s_n + t_n) \leq s + \alpha$, follows from Chapter 3 #5 in Rudin (above.) Therefore we have equality.

(IV)(a) Let $\epsilon > 0$. There exists N such that $n > N \implies |x_n| < \epsilon$. Then for $n > N$,

$$\left| \frac{x_{N+1} + \cdots + x_n}{n} \right| \leq \frac{1}{n} \sum_{k=N+1}^{n} |x_k| \leq \frac{1}{n}(n - N)\epsilon \leq \epsilon.$$

Also $(x_1 + \cdots + x_N)/n \to 0$ as $n \to \infty$, so there exists N_1 such that $n \geq N_1 \implies |x_1 + \cdots + x_N|/n < \epsilon$. Then for $n \geq \max(N, N_1)$,

$$\left| \frac{x_1 + \cdots + x_n}{n} \right| \leq \left| \frac{x_1 + \cdots + x_N}{n} \right| + \left| \frac{x_{N+1} + \cdots + x_n}{n} \right| < 2\epsilon.$$

Since ϵ is arbitrary, this shows $a_n \to 0$.

(b) Take $x_n = (-1)^n$. Then $x_1 + \cdots + x_n$ is either 0 or -1 for all n, so a_n is either 0 or $-1/n$, so $a_n \to 0$, though $x_n \not\to 0$.

(c) We prove the contrapositive. Suppose $\{x_k\}$ is bounded, say $|x_k| \leq M$ for all k. Then $|a_n| = |x_1 + \cdots + x_n|/n \leq (|x_1| + \cdots + |x_n|)/n \leq nM/n = M$ for all n, so $\{a_n\}$ is bounded.

(V) For even n, the sequence is $\left(1 + \frac{1}{n}\right)^n \to e$, and for odd n it is $\left(1 + \frac{1}{n}\right)^{-n} \to 1/e$. Therefore e and $1/e$ are the only subsequential limits, so the lim sup is e and the lim inf is $1/e$.

(VI) Let $p_N \in E$. Then $d(p_N, p) > 0$ (since all points are assumed distinct), so we can take $0 < r < d(p_N, p)/2$. Then the neighborhoods $N_r(p)$ and $N_r(p_N)$ are disjoint. Since $p_n \to p$, there are only finitely many points of E outside $N_r(p)$, hence only finitely many in $N_r(p_N)$. This means that p_N is not a limit point of E, so it is an isolated point.

(VII)(a) $(-\infty, x]$ is a closed set, and $a_n \in (-\infty, x]$ for all n, so $a \in (-\infty, x]$, that is, $a \leq x$.

(b) If $\sup\{a_n\} = \infty$ there is nothing to prove, so assume $y = \sup\{a_n\} < \infty$. For any converging subsequence $a_{n_k} \to a$ we have $a_{n_k} \leq y$ for all k, so $a \leq y$ by (a). Therefore the lim sup (the largest subsequential limit) is bounded by y as well.

(VIII) Suppose $\{x_n\}$ is bounded, say $|x_n| \leq M$ for all n. Given $\epsilon > 0$ there exists N such that $n \geq N \implies |\delta_n| < \epsilon/M \implies |x_n \delta_n| = |x_n||\delta_n| < M \cdot \epsilon/M = \epsilon$. This shows $x_n \delta_n \to 0$.

(IX) Suppose $a_n \to a$. From Chapter 1 #13 we have $||a_n| - |a|| \leq |a_n - a| \to 0$, so $|a_n| \to |a|$.

(X)(a) Since A, B are closed, $A \cap \bar{B} = A \cap B = \phi$ and $\bar{A} \cap B = A \cap B = \phi$. Thus A and B are separated.

(b) Suppose A, B are open and disjoint. If $x \in B$ then x has a neighborhood $N \subset B$ so N contains no points of A. This shows $x \notin A'$. Thus $B \cap A' = \phi$, so $B \cap \bar{A} = \phi$. Similarly $A \cap \bar{B} = \phi$. Thus A, B are separated.

(c) Since B is a neighborhood, it is open. To show A is open, let $x \in A$ and $0 < \delta < d(p, x) - r$. If $y \in N_\delta(x)$ then

$$d(p, x) \leq d(p, y) + d(y, x) < d(p, y) + \delta \quad \text{so} \quad d(y, x) > d(p, y) - \delta > r,$$

so $y \in A$. This shows x has a neighborhood $N_\delta(y)$ in A, so A is open. Since A, B are open and disjoint, by part (b) they are separated.

(d) Let $0 < r < d(p, q)$ and define A, B as in part (c). If there are no points z with $d(p, z) = r$, then $A \cup B$ is all of X, and by part (b), A and B are separated, and nonempty since $p \in B$ and $q \in A$, so X is not connected, a contradiction. Thus there must be a point $z \in X$ with $d(p, z) = r$; this is true for each r between 0 and $d(p, q)$. Since there are uncountably many r values, there must be uncountably many corresponding z's, so X is uncountable.