MATH 425a ASSIGNMENT 5
FALL 2016 Prof. Alexander
Due Wednesday October 12.

Rudin Chapter 3 #3, 5, plus the problems (I)–(X) below:

(I) Suppose $s_n \to 2$, and $t_n \leq 3$ for all n.
(a) If $s_{n_k} + t_{n_k} \to c$ for a subsequence of $\{s_n + t_n\}$, show that $t_{n_k} \to c - 2$.
(b) Show that $\limsup_{n\to\infty}(s_n + t_n) \leq 5$.

(II) Suppose G is open, $p \in G$, and $p_n \to p$. Show that there are at most finitely many points p_n with $p_n \notin G$.

(III) Suppose $\{s_n\}$ and $\{t_n\}$ are bounded sequences in $\mathbb{R}$, $s_n \to s, \alpha = \limsup t_n$, and $\beta = \limsup(s_n + t_n)$. Show that $\beta = s + \alpha$, that is,

$$\limsup(s_n + t_n) = \lim s_n + \limsup t_n.$$

(IV)(a) Show that if $x_k \to 0$ in $\mathbb{R}$ then the averages $a_n = (x_1 + .. + x_n)/n$ also converge to 0.
(b) Disprove the converse of (a) by giving an example.
(c) Show that if $\{a_n\}$ is unbounded then $\{x_k\}$ is unbounded.

(V) Find the lim sup and lim inf of the sequence $\left(1 + \frac{1}{n}\right)^{(-1)^n n}$.

(VI) In a metric space X, suppose $p_n \to p$, all the points p_n and p are distinct, and $E = \{p_n : n \geq 1\}$. Show that every p_n is an isolated point of E.

(VII)(a) Let $A \subset \mathbb{R}$ and $A_x = A \cup \{x\}$. Show that $\sup A_x \geq \sup A$.
(b) Let $\{x_k\}$ be a bounded sequence in $\mathbb{R}$, and $M_n = \sup\{x_n, x_{n+1}, ...\}$. Show that $L = \lim_n M_n$ exists.
(c) In (b), if s is a subsequential limit of $\{x_k\}$, show that $s \leq L$.

(VIII) Suppose $\{x_n\}$ is a bounded sequence in $\mathbb{R}$, and $\delta_n \to 0$. Show that $x_n\delta_n \to 0$.

(IX) Suppose $a_n \to a$ in $\mathbb{C}$. Show that $|a_n| \to |a|$.

(X)(a) Suppose A and B are closed and $A \cap B = \emptyset$. Show that A and B are separated.
(b) Suppose A and B are open and $A \cap B = \emptyset$. Show that A and B are separated.
(c) Let $p \in X$ and $r > 0$, and define $A = \{x \in X : d(x,p) > r\}$ and $B = B_r(p) = \{x \in X : d(x,p) < r\}$. Show that A, B are separated.
(d) Suppose X contains at least one other point $q \neq p$, and X is connected. Show that X is uncountable.

HINTS:

(3) Prove the statement "$s_n < 2$ and $s_n \leq s_{n+1}$" by induction on n. Note in some older printings of the book, the last part of the problem is garbled—it should read, "...and that $s_n < 2$ for $n = 1, 2, 3, \ldots$"

(III) For two subsequences $\{t_{n_k}\}$ and $\{s_{n_k} + t_{n_k}\}$ with the same indices, what happens to the second when the first converges, say to α? Consider also the opposite direction.

(IV)(a) Given $\epsilon > 0$ there exists N such that $n \geq N$ implies $|x_n| < \epsilon$. Deal with $x_1, .., x_{N-1}$ separately.

(c) Try the contrapositive.

(V) You can use the fact from calculus that $\left(1 + \frac{1}{n}\right)^n \to e$.

(VI) Limit points and subsequential limits for E are the same thing. (Why? This is not always true!) Suppose some p_n is a limit point of E and get a contradiction.

(VII)(b) Don't do a "Let $\epsilon > 0$..." proof, instead compare M_n and M_{n+1}.

(X)(a),(b) Use the definition of separated. Also for (b), what property do A^c and B^c have?

(d) For r satisfying $0 < r < d(p, q)$, based on (a)–(c) what happens if there are no points x with $d(p, x) = r$? If there is such an x for every r, what does this tell you about (un)countability of the metric space?