MATH 425a ASSIGNMENT 4 SOLUTIONS
FALL 2016 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 2:

(12) Let $\{G_\alpha, \alpha \in A\}$ be an open cover of K. Since $0 \in K$, we have $0 \in G_{\alpha_0}$ for some α_0. Since G_{α_0} is open, there is a neighborhood $N_\epsilon(0) \subset G_{\alpha_0}$. Since $1/n \to 0$, there exists N such that $n \geq N \implies 1/n \in N_\epsilon(0)$. For each $n = 1, \ldots, N-1$, since $1/n \in K$, there exists G_{α_n} such that $1/n \in G_{\alpha_n}$. Thus $G_{\alpha_0} \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_{N-1}}$ contains 0 and all points $1/n$, that is, $\{G_{\alpha_0}, \ldots, G_{\alpha_{N-1}}\}$ is a finite subcover of K. This shows K is compact.

(22) Let $\mathbb{Q}^k = \mathbb{Q} \times \cdots \times \mathbb{Q}$ be the set of all points of $\mathbb{R}^k$ with rational coordinates. Let $\epsilon > 0$ and $x \in \mathbb{R}^k$. For each coordinate i, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q_i \in \mathbb{Q}$ with $|x_i - q_i| < \epsilon/\sqrt{k}$. Letting $q = (q_1, \ldots, q_k)$ we then have

$$|x - q| = \left(\sum_{i=1}^{k} (x_i - q_i)^2\right)^{1/2} \leq \left(k\frac{\epsilon^2}{k}\right)^{1/2} = \epsilon.$$

This shows $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

Handout:

(A)(i) Let p be a limit point of $N_r(x)$ and let $\epsilon > 0$. Then by definition of limit point, there is a point y of $N_r(x)$ in $N_\epsilon(p)$. Therefore $d(p,x) \leq d(p,y) + d(y,x) < \epsilon + r$. Since ϵ is arbitrary, this shows $d(p,x) \leq r$. Thus both $N_r(x)$ and its limit points are contained in $\{y : d(x,y) \leq r\}$.

(ii) In the metric space $\mathbb{Z}$, we have $N_1(0) = \{0\}$ which is a closed set, so $\overline{N_1(0)} = \{0\}$. But $\{x \in \mathbb{Z} : d(x,0) \leq 1\} = \{-1, 0, 1\}$ so they are not the same.

(B) Since each $x \in E$ is isolated, there exists a radius $r(x)$ such that $E \cap N_{r(x)}(x) = \{x\}$. Since each $x \in N_{r(x)}(x)$, the collection $\{N_{r(x)}(x) : x \in E\}$ forms an open cover of E. Let $\{N_{r(x_1)}(x_1), \ldots, N_{r(x_m)}(x_m)\}$ be any finite subcollection. Then $E \cap \left(\cup_{i=1}^m N_{r(x_i)}(x_i) \right) = \{x_1, \ldots, x_m\}$, which is finite, so it isn't all of E. This means no finite subcollection can cover E, that is, the original collection has no finite subcover. This shows E is not compact.

(C)(i) Let $\{G_\alpha, \alpha \in A\}$ be an open cover of $K \cup \{p\}$. Since this is also an open cover of K, there is a finite subcover of K, say $K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_m}$. There is also some G_β containing p. Then $\{G_{\alpha_1}, \ldots, G_{\alpha_m}, G_\beta\}$ is a finite subcover of $K \cup \{p\}$. Thus $K \cup \{p\}$ is compact.

(ii) We first show that in general, the union of two compact sets L and M is compact. Let $\{G_\alpha, \alpha \in A\}$ be an open cover of $L \cup M$. Since this is also an open cover of each of the compact sets L and M individually, there is a finite subcover of L, say $\{G_\alpha : \alpha \in B\}$, and a finite subcover of M, say $\{G_\alpha : \alpha \in C\}$. Then $\{G_\alpha : \alpha \in B \cup C\}$ is a finite subcover of $L \cup M$. Thus $L \cup M$ is compact.

In this problem, since K is closed, so is $D \cap K$, so $D \cap K$ is a closed subset of a compact set, so $D \cap K$ is compact by 2.35. By assumption $D \cap K^c$ is compact, so by (i), $D = (D \cap K) \cup (D \cap K^c)$ is compact.

(D) Since G_j is open, G_j^c is closed and bounded in $\mathbb{R}$, hence it is compact. Since $G_1^c \supset G_2^c \supset \dots$, it follows from the Corollary after 2.36 that $\cap_{j\geq 1} G_j^c \neq \emptyset$. Therefore $\cup_{j\geq 1} G_j = \left(\cap_{j\geq 1} G_j^c\right)^c \neq \mathbb{R}$.

(E)(i) Compact because it is closed (the only limit point is 0 which is in the set) and bounded (all points are in $[0, 1]$.)

(ii) Not compact because it isn't bounded—it contains points $(x, 1/x)$ for arbitrarily large x.

(iii) Compact because it is closed and bounded.

(F) One example is $\{(1/n, 1) : n \geq 2\}$. If $x \in (0, 1)$ then $x > 1/n$ for some n, so $x \in (1/n, 1)$, so $x \in \cup_{n\geq 1}(1/n, 1)$. It follows that the sets $\{(1/n, 1) : n \geq 2\}$ cover $(0, 1)$, and they are open, so they are an open cover. But if we take any finite subcollection $(1/n_1, 1), \dots, (1/n_m, 1)$ with $n_1 < \cdots < n_m$, then the union of this subcollection is $(1/n_m, 1)$ so it does not cover $(0, 1)$. Thus our open cover has no finite subcover.

(G) $[-1, 1]$ is closed in $\mathbb{R}$, and $E = (0, 1] = [-1, 1] \cap Y$, so E is closed relative to Y by Theorem 2.30. E is bounded since $d(x, 1) \leq 1$ for all $x \in E$. But E is not compact in $\mathbb{R}$, by Theorem 2.34, since it isn't closed in $\mathbb{R}$, so E is also not compact in Y, by Theorem 2.33.