MATH 425a ASSIGNMENT 4
FALL 2016 Prof. Alexander
Due Friday September 30.

Note this assignment is due after Midterm 1, but the material IS covered on the midterm.

Rudin Chapter 2 #12, 22 plus the problems (A)–(G) below:

(A)(i) Show that in any metric space, the closure of a neighborhood satisfies $\overline{N_r(x)} \subset \{y : d(x,y) \leq r\}$.
(ii) Find an example where the sets in (i) are not equal.

(B) Suppose E is infinite and all points of E are isolated. Show *directly from the definition* (of compact) that E is not compact. This means do it using open covers, not by citing some theorem.

(C)(i) Suppose K is compact and $p \notin K$. Show directly from the definition of compactness that $K \cup \{p\}$ is compact.
(ii) Show that if K is compact, D is closed, and $D \cap K^c$ is compact, then D is compact.

(D) Suppose $G_1 \subset G_2 \subset \dots$ are open in $\mathbb{R}$, and G_j^c is nonempty and bounded for all j. Show that $\cup_{j \geq 1} G_j \neq \mathbb{R}$.

(E) Identify which of the following sets are compact and which are not, with an explanation (not necessarily a full formal proof.)
(i) $\{1/k : k \in \mathbb{N}\} \cup \{0\}$
(ii) $\{(x,y) \in \mathbb{R}^2 : |xy| \leq 1\}$
(iii) $[2,3] \cup [4,5]$

(F) Find an open cover of $(0,1)$ which does not have a finite subcover.

(G) Show that in the metric space $Y = (0,2)$, the set $E = (0,1]$ is closed and bounded, but E is not compact.

HINTS:

(A)(ii) You can use $\mathbb{Z}$ as your metric space.

(C) For (i), use the definition of compactness. For (ii), use (i).

(D) This is closely related to one of the textbook's theorems about compact sets.

(F) Suppose you have a collection of *finitely* many open intervals, each having left endpoint strictly > 0. How can you find h such that none of $(0,h]$ is covered by your intervals? Could your finite collection be an open cover of $(0,1)$? How can you define an *infinite* collection of open intervals, each having left endpoint strictly > 0, which together do form a cover of $(0,1)$?

(G) Theorem 2.33 may be useful.

(22) Let $\epsilon > 0$ and $x, y \in \mathbb{R}^k$. Fill in the blank: if the coordinates satisfy $|y_i - x_i| <$ ____ for all i, then $|y - x| < \epsilon$. Now use this together with the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.