MATH 425a ASSIGNMENT 3 SOLUTIONS
FALL 2016 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 2:

(9)(a) Suppose $x \in E^\circ$. This means there is a neighborhood $N_r(x) \subset E$. By 2.19, $N_r(x)$ is open, so given $y \in N_r(x)$ there is a radius s such that $N_s(y) \subset N_r(x) \subset E$. This shows that $y \in E^\circ$, and then since $y \in N_r(x)$ is arbitrary, we get that $N_r(x) \subset E^\circ$. Therefore E° is open.

(c) Suppose $G \subset E$ and G is open. If $x \in G$, then x has a neighborhood contained in G, so this neighborhood is also contained in E. This means $x \in E^\circ$. Thus $G \subset E^\circ$.

Handout:

(I) $\left\{\frac{1}{n}, 1+\frac{1}{n}, 2+\frac{1}{n}, 3+\frac{1}{n} : n \geq 2\right\}$ is one example. Its limit points are $\{0,1,2,3\}$.

(II)(a) If E is open and $x \in E$, there is a neighborhood $N_r(x) \subset E$. For all $s \leq r$, $N_s(x)$ contains points other than x and all these points are in E. For $s > r$, $N_s(x)$ contains $N_r(x)$ which in turn contains points other than x, and again these points are in E. This shows all neighborhoods of x contain points of E other than x, so x is a limit point of E.

(b) Given $n \in \mathbb{Z}$, the neighborhood $(n-\frac{1}{2}, n+\frac{1}{2})$ contains no point of $\mathbb{Z}$ other than n itself. Thus n is not a limit point of $\mathbb{Z}$.

(III) d_1 is not a metric. For example, consider the points $0, 1, 2$. We have $d_1(0,1) = d_1(1,2) = 1$ but $d_1(0,2) = 8$, so $d_1(0,2) > d_1(0,1) + d_1(1,2)$ and the triangle inequality fails.

d_2 is not a metric. For $x \in \mathbb{R}$ we have $d_2(x, x-1) = 0$ but $x \neq x-1$.

d_3 is a metric. It is clear that $d_2(x,x) = 0$ and $d_2(x,y) = d_2(y,x)$ for all x, y. To prove the triangle inequality, first observe that for $a, b \geq 0$ we have $(\sqrt{a}+\sqrt{b})^2 = a + 2\sqrt{ab} + b \geq a + b$ so $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. Using this and the triangle inequality for Euclidean distance, we see that for $x, y, z \in \mathbb{R}$,

$$\sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|},$$

which shows the triangle inequality holds for d_2.

(IV)(a) Suppose x is not a limit point of any A_i. Then for each $i \leq n$, there is a radius $r_i > 0$ such that $N_{r_i}(x)$ contains no points of A_i except possibly x itself. Let $r = \min\{r_i : i \leq n\}$. Then $N_r(x)$ contains no points of $\cup_{i=1}^n A_i$, except possibly x itself. Thus x is not a limit point of $\cup_{i=1}^n A_i$.

(b) Part (a) says "if $x \in (\cup_{i=1}^n A_i')^c$ then $x \in (B')^c$, or equivalently, $(\cup_{i=1}^n A_i')^c \subset (B')^c$. Taking complements shows this is also equivalent to $B' \subset \cup_{i=1}^n A_i'$.

(c) From (b) and the definition of B we have $B \cup B' \subset (\cup_{i=1}^n A_i) \cup (\cup_{i=1}^n A_i') = \cup_{i=1}^n (A_i \cup A_i')$, or equivalently, $\overline{B} \subset \cup_{i=1}^n \overline{A}_i$.

(V) Choose $r, s > 0$ with $r + s \leq d(x, y)$. Then for $z \in N_r(x)$, from the triangle inequality we have $d(z, y) \geq d(x, y) - d(x, z) \geq d(x, y) - r \geq s$, so $z \notin N_s(y)$. This shows $N_r(x) \cap N_s(y) = \emptyset$.

(VI) For each point $x \in F$, x is isolated so there is a radius $\delta_x > 0$ such that $N_{\delta_x}(x) \cap F = \{x\}$. This mean that for $x, y \in F$ we have $d(x, y) \geq \delta_x$ and $d(x, y) \geq \delta_y$, and therefore $(*)$ $d(x, y) \geq (\delta_x + \delta_y)/2$. The neighborhoods $N_{\delta_x}(x)$ are not necessarily disjoint, but we can let $r_x = \delta_x/2$. By $(*)$, we then have $d(x, y) \geq r_x + r_y$ for all x, y in F with $x \neq y$. By the solution of problem (II), this shows $N_{r_x}(x)$ and $N_{r_y}(y)$ are disjoint for all $x \neq y$ in F.

(VII) Since $A \subset F$ and F is closed, we have $A' \subset F' \subset F$, so $\overline{A} = A \cup A' \subset F$.

(VIII)(a) $E_n = [n, \infty)$ is one example.

(b) $E_1 = \emptyset, E_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ for $n \geq 2$ is one example. The union is $(0, 1)$ which is not closed.