MATH 425a ASSIGNMENT 3
FALL 2016 Prof. Alexander
Due Wednesday September 21.

Rudin Chapter 2 #7b, 9ac, plus the problems (I) - (VIII) below:

(I) Find a subset of $\mathbb{R}$ which has exactly 4 limit points.

(II)(a) Let G be an open subset of $\mathbb{R}$. Show that every point $p \in E$ is a limit point of E.

(b) Show that in the metric space $\mathbb{Z}$ (with the usual distance), no point $n \in \mathbb{Z}$ is a limit point of $\mathbb{Z}$.

(III) Define the following functions on $\mathbb{R} \times \mathbb{R}$:

$$d_1(x,y) = |x-y|^3, \qquad d_2(x,y) = |x-(y+1)|, \quad d_3(x,y) = \sqrt{|x-y|}.$$

Determine whether each one is a metric or not.

(IV) Suppose $A_1, \dots A_n$ are subsets of a metric space.

(a) If x is not a limit point of any of the sets A_i (that is, $x \notin \cup_{i=1}^n A_i'$), show that x is not a limit point of $\cup_{i=1}^n A_i$.

(b) Let $B = \cup_{i=1}^n A_i$. Restate (a) as a statement about the relation between B' and $\cup_{i=1}^n A_i'$. Justify, at least informally.

(c) Show that $\overline{B} \subset \cup_{i=1}^n \overline{A_i}$.

(V) Suppose x, y are two distinct points of a metric space. Show that there exist radii $r, s > 0$ such that the neighborhoods $N_r(x)$ and $N_s(y)$ are disjoint. How do the values r, s and $d(x,y)$ have to be related, in order for your proof to work?

(VI) Suppose X is a metric space, and every point of some set $F \subset X$ is an isolated point of F. Show that you can choose a neighborhood $N_{r_x}(x)$ of each $x \in F$ such that none of the $N_{r_x}(x)$'s overlap, that is, $N_{r_x}(x) \cap N_{r_y}(y) = \phi$ for all $x, y \in F$.

Here r_x is some radius that may be different for different x's.

(VII) Suppose $A \subset F$ and F is closed. Show that $\overline{A} \subset F$.

(VIII)(a) Give an example of nonempty closed sets $E_1 \supset E_2 \supset \dots$ such that $\cap_{n \geq 1} E_n = \emptyset$.

(b) Give an example of nonempty closed sets $E_1 \subset E_2 \subset \dots$ such that $\cup_{n \geq 1} E_n$ is open.

HINTS:

(9)(a) If $x \in E^\circ$, then by definition, $N_r(x) \subset E$ for some r. Show that all points $y \in N_r(x)$ are interior points of E. See Theorem 2.19.

(I) First construct a set with just one limit point, say 0.

(III) d_3 is the tricky one. First prove this general fact about nonnegative numbers: $\sqrt{b+c} \leq \sqrt{b}+\sqrt{c}$. (What operation can you perform on both sides to simplify this inequality?) Then relate this to the triangle inequality.

(V) Draw a picture of two points x, y and their disjoint neighborhoods, in the plane. Use this to help you understand how r, s and $d(x, y)$ have to be related.

(VI) This is a more difficult one. Each x being isolated means each x has a neighborhood containing no other points of x. But unless we do more, these neighborhoods might overlap. How can you specify the radii of these balls, so that they *don't* overlap? See problem (II).

(VII) This is very quick, really just one or two sentences.

(VIII) You can use (possibly unbounded) intervals in $\mathbb{R}$.