MATH 425a ASSIGNMENT 2 SOLUTIONS
FALL 2016 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 1:

(17)

$$\begin{aligned}
|x+y|^2+|x-y|^2 &= (x+y)\cdot(x+y)+(x-y)\cdot(x-y)\\
&= (x\cdot x+y\cdot x+x\cdot y+y\cdot y)+(x\cdot x-y\cdot x-x\cdot y+y\cdot y)\\
&= 2x\cdot x+2y\cdot y\\
&= 2|x|^2+2|y|^2.
\end{aligned}$$

To interpret this, consider the parallelogram with vertices $0, x, y, x+y$. Its diagonals have lengths $|x-y|, |x+y|$, so the equality says the sum of the squares of the two diagonals is the sum of the squares of the four sides.

Chapter 2:

(3) Since the set $\mathbb{A}$ of algebraic real numbers is countable and $\mathbb{R}$ is not, we have $\mathbb{A} \neq \mathbb{R}$, so some real numbers aren't in $\mathbb{A}$.

Handout:

(A) If $z \in A$ then $|z+1| \leq |z|+1 \leq \alpha+1$. This says that $\alpha+1$ is an upper bound for the set $E = \{|z+1| : z \in A\}$. Since the sup is the least upper bound, this means $\sup E \leq \alpha+1$.

(B)(a) A_3 is the Cartesian product of countable sets, so it is countable. B_3 is an infinite subset of A_3 so it is also countable, by 2.8.

(b) f is not 1-to-1 since $f\big((a,b,c)\big) = f\big((b,a,c)\big)$. f is onto, since given a set $\{a,b,c\}$, we can put its elements in some order to make a tuple in B_3, say $(a,b,c) \in B_3$, and then $f\big((a,b,c)\big) = \{a,b,c\}$.

(c) B_3 is countable by (a), and $C_3 = f(B_3)$ by (b), so by a theorem from lecture, C_3 is at most countable. Since C_3 is not finite, it must be countable.

(d) Let $C_n = \{\text{all } n\text{-element subsets of } \mathbb{Z}\}$. The same argument as the above for C_3 shows that C_n is countable. Since $C = \cup_{n=1}^{\infty} C_n$, it follows from 2.12 that C is countable.

(C) For the intersection, consider $[0,1]$ and $[1,2]$. These are uncountable but the intersection is the single point $\{1\}$ which is not uncountable.

For the union, suppose A, B are uncountable. If $A \cup B$ were finite, then its subsets A and B would be finite, a contradiction. If $A \cup B$ were countable, then A would be an infinite subset of the countable set $A \cup B$, hence countable, again a contradiction. Therefore $A \cup B$ must be uncountable.

(D)(a) Let A_N be the set of sequences which are 0 after time N, that is,

$$A_N = \{(z_1, z_2, \dots) : z_n \in \{0, 1, 2, 3\} \text{ for all } n, z_n = 0 \text{ for all } n > N\}.$$

Then A_N is finite (in fact it has 4^N elements), since an element of A_N is determined by specifying each of the first N coordinates, with 4 choices for each coordinates. The set $A = \{$ all terminating sequences of 0's, 1's, 2's, and 3's $\}$ is the same as $\cup_{N \geq 1} A_N$, so by the Corollary to Theorem 2.12, A is at most countable. Since A is infinite, it must be countable.

(b) Similarly to (a), we can let B_N be the set of sequences which are 0 after time N, that is,

$$B_N = \{(z_1, z_2, \dots) : z_n \in \mathbb{Z} \text{ for all } n, z_n = 0 \text{ for all } n > N\}.$$

We can make a bijection between B_N and $\mathbb{Z}^N$, defining $f : \mathbb{Z}^N \to B_N$ by $f(z_1, \dots, z_N) = (z_1, \dots, z_N, 0, 0, \dots)$. By Theorem 2.13, $\mathbb{Z}^N$ is countable; since we have a bijection, so is B_N. The set $B = \{$ all terminating sequences of integers $\}$ is the same as $\cup_{N \geq 1} B_N$, which is countable by Theorem 2.12.

(E) Since $\mathbb{C}$ has the same metric as $\mathbb{R}^2$, we know the triangle inequality: $|w_1 + w_2| \leq |w_1| + |w_2|$. So the inequality is valid for the starting value $n = 2$. We can proceed by induction. Suppose the inequality

$$(*) \qquad |w_1 + \cdots + w_n| \leq |w_1| + \cdots + |w_n|$$

is true for some $n \geq 2$. From (*) for two complex numbers, we know that

$$|w_1 + \cdots + w_n + w_{n+1}| \leq |w_1 + \cdots + w_n| + |w_{n+1}|.$$

Then from (*) for n complex numbers, we get

$$|w_1 + \cdots + w_n| + |w_{n+1}| \leq |w_1| + \cdots + |w_n| + |w_{n+1}|,$$

so (*) is true for $n + 1$. Thus by induction it is true for all n.

(F) To show $|a| \leq |x - y|$ for some a, we show $a \leq |x - y|$ and $-a \leq |x - y|$. In the present case, a is $|x| - |y|$. From the triangle inequality we have

$$|x| \leq |x - y| + |y| \quad \text{so} \quad |x| - |y| \leq |x - y|,$$
$$|y| \leq |y - x| + |x| \quad \text{so} \quad |y| - |x| \leq |x - y|.$$

Putting these together shows $\big||x| - |y|\big| \leq |x - y|$.

(G) Note that $A \backslash B = \{x \in A : x \text{ is rational}\}$, which is at most countable since it's a subset of $\mathbb{Q}$. If B were at most countable, then $A = B \cup (A \backslash B)$ would be the union of two at-most-countable sets, contradicting the assumed uncountability of A. Thus B must be uncountable.