MATH 425a ASSIGNMENT 2
FALL 2016 Prof. Alexander
Due Wednesday September 14.

Rudin Chapter 1 #17, Chapter 2 #3 plus the problems (A) - (G) below:

(A) Let $A \subset \mathbb{C}$ and $\alpha = \sup\{|z| : z \in A\}$. Show that $\sup\{|z+1| : z \in A\} \le \alpha + 1$.

(B) Let

$$\begin{aligned} A_3 &= \{(a,b,c) : a,b,c \in \mathbb{Z}\} \\ B_3 &= \{(a,b,c) \in A_3 : a \neq b, a \neq c, b \neq c\} \\ C_3 &= \{\text{all 3-element subsets of } \mathbb{Z}\}. \end{aligned}$$

Define $f : B_3 \to C_3$ by $f\big((a,b,c)\big) = \{a,b,c\}$. Note that in the vector (a,b,c) there is ordering: (c,a,b) is not the same vector. In contrast, $\{a,b,c\}$ is just a set of 3 elements; there is no ordering.

(a) Show that A_3 and B_3 are countable.
(b) Is f 1-to-1? Onto? Explain.
(c) Show C_3 is countable.
(d) Show that $C = \{\text{all finite subsets of } \mathbb{Z}\}$ is countable.

(C) Is the intersection of two uncountable sets necessarily uncountable? How about their union? (Prove, or disprove by giving an example. This is short!)

(D) Let us say that a sequence $\{z_n\}$ of integers *terminates* if for some $N, z_n = 0$ for all $n \ge N$. Thus for example $(1,3,0,3,0,0,0,...)$ and $(1,2,1,3,1,2,0,0,0,...)$ both terminate.

(a) Show that $A = \{$ all terminating sequences of 0's, 1's, 2's, and 3's $\}$ is countable.
(b) Show that $B = \{$ all terminating sequences of integers $\}$ is countable.

(E) Show that for all n and all complex numbers $w_1, \ldots, w_n$,

$$(*) \quad |w_1 + \cdots + w_n| \le |w_1| + \cdots + |w_n|.$$

(F) Show that $|\, |z| - |w| \,| \le |z - w|$ for all complex numbers z, w.

(G) Suppose $A \subset \mathbb{R}$ is uncountable. Show that $B = \{x \in A : x \text{ is irrational}\}$ is uncountable.

HINTS:

(For the assignment in general) Remember these two useful facts: (i) To show $|a| \le b$, show $a \le b$ and $-a \le b$. (ii) To prove inequalities with norms, it is sometimes useful to use $|x|^2 = x \cdot x$.

(3) You may *assume problem 2* in Rudin to do this. Then all you need is that (from lecture and problem 2) the algebraic numbers are countable, but the reals are not.

(B)
(D)(a) What happens if you fix the termination time? By this I mean the index of the last non-zero entry. For the two examples given in the problem, the termination times are 4 and 6.

(E) Use induction. For $n = 2$, $(*)$ comes directly from one of the theorems of Chapter 1. Suppose $(*)$ is true for all sums of length up to n, and prove it from this for sums of length $n + 1$.

(G) Prove the contrapositive.