# MATH 425a ASSIGNMENT 1 SOLUTIONS
FALL 2016 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 1:

(4) Let $x$ be any element of $E$. Since $\alpha$ is a lower bound and $\beta$ is an upper bound, we must have $\alpha \leq x \leq \beta$, so $\alpha \leq \beta$.

Handout:

(I)(a) True (take any $y < 0$.)
(b) True (take any $x > 0$.)
(c) False (the statement fails for $y < -x^2$.)
(d) False (the statement fails for $a = b = 0, c = 1$, for example.)

(II)(a) For every number, there is a number smaller than the square of the first number.
(b) For some number, adding any square to it gives a positive result.
(c) There exists a number such that if add any second number to its square, the result is positive.
(d) Every polynomial of degree at most two has a real root.

(III) It is equivalent to (a). The negation of the original statement "$\forall x\ \exists y : p(x, y)$ is true" is obtained by reversing the quantifiers, so it is given by the statement in parentheses in (a). Thus the negation of the statement in parentheses in (a) is the negation of the negation of the original statement, which is the same as the original statement.

(IV)(a) $\{2, 4, 8\}$
(b) $\{2, 4, 6, 8, 10, 12, 14, 16, 32\}$
(c) $\{x \in \mathbb{R} : 0 < x < 9\}$
(d) $\{2, 4, 6, 8, 10, 12, 14\}$

(V) Note $45 = 3^2 \cdot 5$. If $\sqrt{45} = p/q$ in lowest terms, then (*) $p^2 = 3^2 \cdot 5 \cdot q^2$, so 5 divides $p^2$, so 5 divides $p$. But then $5^2$ divides $p^2$, so from (*), 5 must divide $q^2$, so 5 divides $q$. But this means $p/q$ is not in lowest terms, a contradiction. Thus $\sqrt{45}$ must be irrational.

Note this argument doesn't work when a prime number has an even power, like the factor $3^2$ in 45. So we based our argument around the prime 5, instead.

(VI) The problem statement should have referred to problem (V), not to Rudin Chapter 1 #2. The thing that makes (V) work is that in the number $n = 45 = 3^2 \cdot 5$, the 5 is not a square, so when 5 divides $p$, it forces 5 to divide $q$ as well. To have a non-square factor and do basically the same proof, we need some prime to appear with an odd power when we factor $n$. But this just means $n$ itself is not a square.

Here is a more formal proof. We can equivalently prove the contrapositive: if $\sqrt{n}$ is rational, then $n$ must be a square, that is, $\sqrt{n}$ must be an integer. So suppose $\sqrt{n} = p/q$ is rational and expressed in lowest terms. This means $p^2 = q^2 n$. Since they are squares, every prime factor in $p^2$ or $q^2$ appears with an even exponent. If some prime $k$ appears with an odd exponent in $n$, then since $k$ has an even exponent (possibly 0) in $q^2$, it has an odd exponent in $q^2 n$, that is, in $p^2$. But no primes have odd exponents in $p^2$, since it's a square, so there can be no such $k$. Thus every prime $k$ appears with an even exponent (possibly 0) in $n$, which makes $n$ a square.

(VII) If $q - x$ were rational, then $(q - x) - q = -x$ would also be rational, hence $x$ would be rational, a contradiction. Therefore $q - x$ is irrational.

If $x/q$ were rational then the product $q \cdot (x/q) = x$ would be rational, a contradiction. Therefore $x/q$ is irrational.

(VIII) Let $\alpha = \inf A$. We want to show that $\sup(-A) = -\alpha$.

First we show $-\alpha$ is an upper bound. If $x \in -A$, then $-x \in A$ so $\alpha \leq -x$, so $-\alpha \geq x$. This shows $-\alpha$ is indeed an upper bound for $-A$.

To show $-\alpha$ is the *least* upper bound, we show every $\gamma < -\alpha$ is *not* an upper bound. Suppose $\gamma < -\alpha$. Then $-\gamma > \alpha = \inf A$, so $-\gamma$ is not a lower bound for $A$, meaning there exists $y \in A$ with $y < -\gamma$. Then $-y > \gamma$ and $-y \in -A$, so $\gamma$ is not an upper bound for $-A$. Thus $-\alpha$ is the least upper bound for $-A$.