MATH 425a MIDTERM EXAM 2 SOLUTIONS
Fall 2016
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(1)(a) (See text)

(b) $\lim_{x\to p} f(x) = q$ means for every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in E, 0 < d(x,p) < \delta \implies d(f(x), f(p)) < \epsilon$.

(2)(a)

$$\left|(-1)^n \frac{2.5^n}{n^{2.5}}\right|^{1/n} = \frac{2.5}{(n^{1/n})^{2.5}} \to 2.5,$$

so $R = 1/2.5 = 0.4$.

(b) Since $a_n \to 0$, $\{a_n\}$ is bounded: there exists M such that $|a_n| \le M$ for all n. Hence $|a_n|/n^p \le M/n^p$, and $\sum \frac{M}{n^p}$ converges since $p > 1$, so $\sum a_n/n^p$ converges by the comparison test.

(c) $3^n/4^n \to 0$ so there exists N such that

$$n \ge N \implies 3^n < \frac{1}{2}4^n \implies 0 < \frac{1}{4^n - 3^n} < \frac{1}{4^n - \frac{1}{2}\cdot 4^n} = \frac{2}{4^n}.$$

Since $\sum 2/4^n$ converges, $\sum 1/(4^n - 3^n)$ converges by the comparison test.

(d) Let $a_n = \sin n - \sin(n-1)$. In the sum $\sum_{k=1}^n (\sin k - \sin(k-1))$, all terms cancel except $\sin n$ and $\sin 0$: the sum is $\sin n - \sin 0 = \sin n$. Therefore the partial sums of $\sum_n a_n$ are bounded. Also $1/n \searrow 0$, so by Theorem 3.42, $\sum \frac{1}{n} a_n$ converges.

(e) Let $a_n = (\frac{2}{3})^{\log_2 n}$. Use Cauchy Condensation Test: $a_{2^k} = (\frac{2}{3})^k$ so $\sum_k 2^k a_{2^k} = \sum_k 2^k (\frac{2}{3})^k = \sum_k (\frac{4}{3})^k$ which diverges since $(\frac{4}{3})^k \not\to 0$. Therefore $\sum_n a_n$ diverges.

(3)(a) $\{p_n\}$ is also a sequence is the compact set $\overline{F}$, so it has a subsequence converging in $\overline{F}$. Therefore this subsequence is Cauchy.

(b) FIRST SOLUTION: Let $E = \{x_n : n \ge 1\}$ and $F = \{f(x_n) : n \ge 1\}$ (both viewed as sets, not sequences.) Since x is a limit point of E, there is a subsequence $x_{n_k} \to x$, by a proposition from lecture. By continuity of f, we have $f(x_{n_k}) \to f(x)$. Since all x_n's are distinct, x cannot appear infinitely often in the sequence $\{x_n\}$; since f is one-to-one, this means $f(x)$ cannot appear infinitely often in the sequence $\{f(x_n)\}$. Therefore by the same proposition, $f(x)$ is a limit point of F.

SECOND SOLUTION: Let $\epsilon > 0$. There exists $\delta > 0$ such that $d(x_n, x) < \delta \implies d(f(x_n), f(x)) < \epsilon$. Since all x_n's are distinct, there is at most one n_0 with $x_{n_0} = x$. Since x is a limit point of $\{x_n : n \ge 1\}$, this means we can find an $n > n_0$ with $0 < d(x_n, x) < \delta$, and since f is one-to-one, this means $0 < d(f(x_n), f(x)) < \epsilon$. Since ϵ is arbitrary, this means $f(x)$ is a limit point of $\{f(x_n) : n \ge 1\}$.

(4)(a) Given $\epsilon > 0$, let $\delta = (\epsilon/C)^{1/\alpha}$. Then $|x - y| < \delta \implies |f(x) - f(y)| \le C|x-y|^\alpha \le C\delta^\alpha = \epsilon$. This shows f is uniformly continuous.

(b) f is continuous at each $p \in E$ (note $0 \notin E$), meaning f is continuous on E.

For uniform continuity, let $0 < \epsilon < 1$. Given $0 < \delta < 1$, choose x, y with $-\frac{\delta}{2} < x < 0$ and $0 < y < \frac{\delta}{2}$. Then $|y - x| < \delta$, but

$$|f(y) - f(x)| = |(y + 1) - (x + 3)| = |y - x - 2|.$$

Provided we choose x, y close enough to 0, we have $|y - x - 2| > 1 > \epsilon$. Thus no $0 < \delta < 1$ works for all x, y, so no $\delta > 0$ works, meaning f is not uniformly continuous.