The frog model on trees

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Joint work with Christopher Hoffman and Matthew Junge

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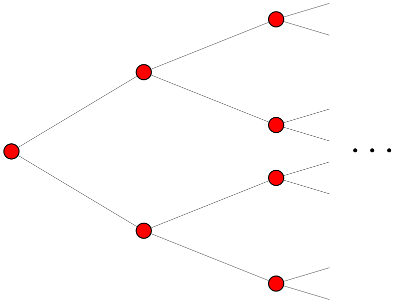
The frog model



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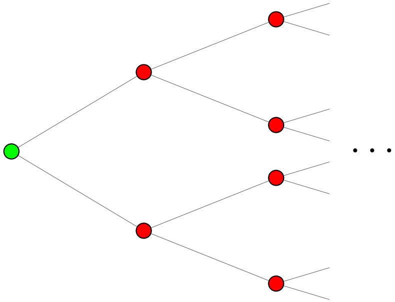

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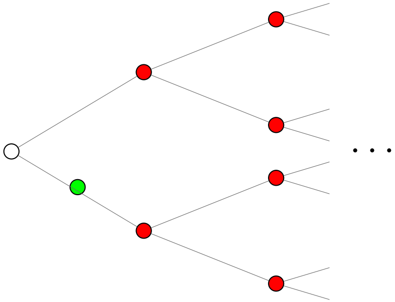

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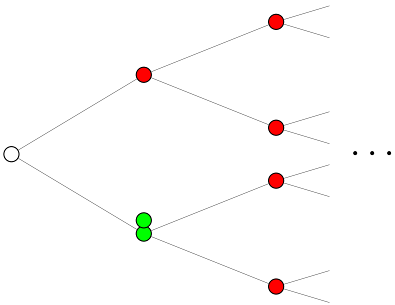

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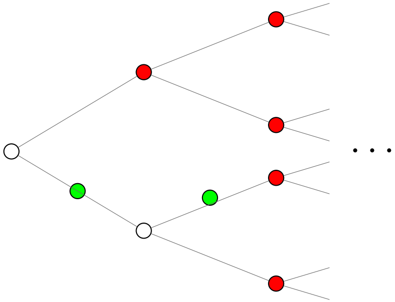

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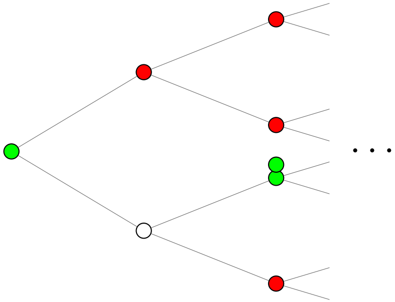

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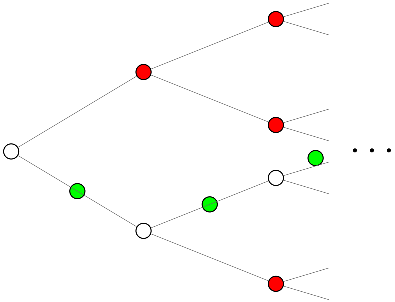

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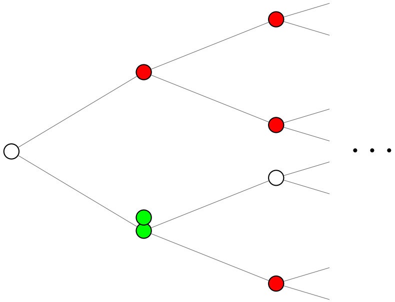

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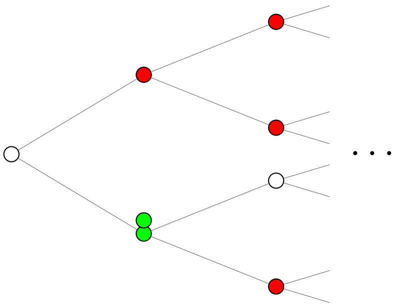


- frogs wake sleeping frogs
- once awake, frogs do independent RWs

Self-interacting random walks

activated random walk Rolla–Sidoravicius 2012

excited random walk Benjamini–Wilson 2003

Main question

Frog model on G: recurrent or transient?

Previous frog results

Theorem (Telcs–Wormald 1999)

Frog model on $\mathbb{Z}^d$*: recurrent w.p. 1*

Theorem (Alves–Machado–Popov 2002, Ramírez–Sidoravicius 2003)

Set of visited vertices in $\mathbb{Z}^d$ *has limiting shape.*

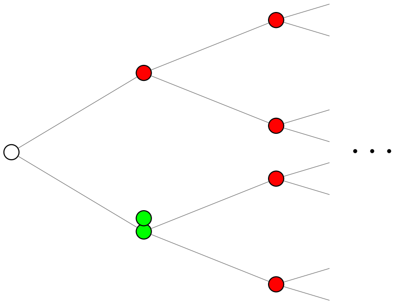
Main result

Theorem (Hoffman–J.–Junge 2014)

Frog model on d-ary tree is

- *recurrent for* $d = 2$
- *transient for* $d \geq 5$

Proof of transience for $d \geq 6$



Proof of transience for $d \geq 6$

weight function: $W_n = \sum_{\substack{f \in \text{frogs} \\ \text{at time } n}} e^{-\theta \,\text{level}(f)}$

Claim. $\mathbf{E}[W_{n+1} \mid W_n] \leq \left(\frac{1}{d+1} e^{\theta} + 2 \frac{d}{d+1} e^{-\theta} \right) W_n$

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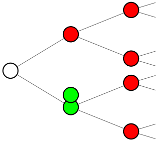
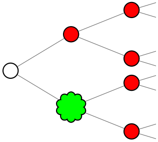
Transience for $d = 5$

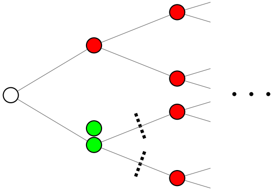
Replace  $\cdots$ by  $\cdots$

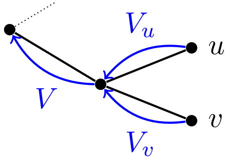
- 27 particle types
- computer-assisted

Recurrence for $d = 2$

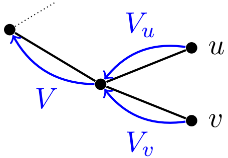
lower bounding process:

- non-backtracking frogs, frozen at root
- ≤ 1 frog enters a subtree:





$$V = \mathrm{Bin}\big(V_v, \tfrac{1}{2}\big) + \mathbf{1}_{\{u \text{ visited}\}}\mathrm{Bin}\big(V_u, \tfrac{1}{2}\big) + \mathrm{Bernoulli}\big(\tfrac{1}{3}\big)$$



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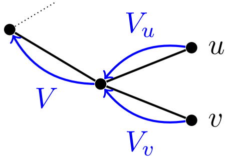
Idea. Assume $V_u, V_v \succ \mathrm{Poi}(\lambda)$, show $V \succ \mathrm{Poi}(\lambda + \epsilon)$



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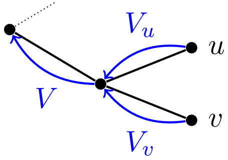
Definition. $\mathcal{A}g(x) = \frac{x+2}{3} g\big(\frac{x+1}{2}\big) + \frac{x+1}{3} g\big(\frac{x}{2}\big)\Big(1 - g\big(\frac{x+1}{2}\big)\Big)$.

Lemma. Let $f(x) = \mathbf{E}x^V$. Then $\mathcal{A}f = f$.

Lemma. If $f(x) \le e^{\lambda(x-1)}$, then $\mathcal{A}f(x) \le e^{(\lambda+\epsilon)(x-1)}$.

Conclusion. $f(x) \equiv 0$, and $V = \infty$ a.s.



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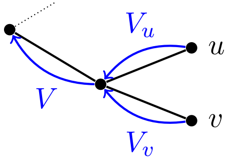

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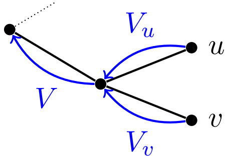

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Main result

Theorem (Hoffman–J.–Junge 2014)

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Frog model with Poi(μ) frogs

Theorem (Hoffman–J.–Junge 2014?)

Frog model on d-ary tree with Poi(μ) *frogs per vertex:*

- *recurrent if* $\mu > \mu_c(d)$
- *transient if* $\mu < \mu_c(d)$

Critical value: $d \ll \mu_c(d) \ll d \log d$

Further questions

Conjecture

Frog model on d-ary tree, one frog per vertex:

- *strongly recurrent for* $d = 2$
- *weakly recurrent for* $d = 3$
- *transient for* $d = 4$

Further questions

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- *strongly recurrent for* $d = 2$
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Conjecture

Frog model on d-ary tree, $\mathrm{Poi}(\mu)$ *frogs per vertex:*

- *3-phase transition in* μ

Further questions

Summary:

- Frog model on $\mathbb{Z}^d$: always recurrent
- Frog model on d-ary trees: has phase transition

Question. On other graphs? Amenable vs. nonamenable?

Further questions

Question (Itai Benjamini). Frog cover time on d-ary tree, height n:

- polynomial in n?
- exponential in n?
- one or the other, depending on d?

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Thanks!