Zeros of the derivatives of random polynomials and random entire functions

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Overview

Random Polynomials of degree n

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where c_j's are independent random variables.
- Another class of random polynomials arise as the characteristic polynomials of random matrices.
- Traditionally, we ask questions about the zeros of these random polynomials.

Random Polynomials - History

- Mark Kac (1943) showed that, if the coefficients c_j's are IID standard normal (real), then the expected number of real zeros are $\sim \frac{2}{\pi} \log n$.

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- More recently, Kabluchko and Zaporozhets (2012) derived the asymptotic empirical distribution of the zeros of elliptic random polynomials and Littlewood-Offord random polynomials under a mild condition.

Random Analytic Functions

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- Gaussian analytic functions on some domain Λ: A function $\mathbf{f}$ such that $(\mathbf{f}(z_1), \cdots, \mathbf{f}(z_n))$ has a mean zero complex Gaussian distribution for every $n \geq 1$ and every $z_1, \cdots, z_n \in \Lambda$.

Random Analytic Functions - History

- The number of zeros of $f(z) = \sum a_n z^n$, where a_n are i.i.d. complex Gaussian, in a disk of radius r about the origin has the same distribution as the sum of independent $\{0, 1\}$-valued random variables X_k, where $\mathbb{P}(X_k = 1) = r^{2k}$ - Peres and Virág (2005).

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- Kabluchko and Zaporozhets (2012) derived the asymptotic empirical distribution of the zeros of flat analytic functions and hyperbolic analytic functions.

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Zeros and Critical Points and Random Functions

QUESTION - A new class of random polynomials and random entire functions

Suppose, instead of the traditional approach of having a probability measure on the coefficients of the polynomial / power series, we now consider probability measures on the zeros.

Can we say anything about the location or distribution of critical points of these functions?

What about their coefficients?

Our Random Polynomials

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Random polynomial of degree n

$$p_n(z) = (z - Z_1)(z - Z_2)...(z - Z_n),$$

where Z_j's are independent random variables.

Our Random Analytic Functions

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If $\{Z_n\}_n$ is a sequence of random variables (assume that, with probability 1 there are no cluster points), then we can use Weierstrass Product Formula to construct an entire function which vanishes at exactly these points.

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Our Random Analytic Functions

If $\{Z_n\}_n$ is a sequence of random variables (assume that, with probability 1 there are no cluster points), then we can use Weierstrass Product Formula to construct an entire function which vanishes at exactly these points.

But there is a small problem: there are infinitely many entire functions that have the same zeros!

Critical points of a complex polynomial with I.I.D zeros

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Let $y_1^{(n)}, y_2^{(n)}, \ldots, y_{n-1}^{(n)}$ be the critical points of p_n and $\mathcal{Z}(f)$ be the empirical distribution of the roots of a function f. Thus,

$$\mathcal{Z}(p_n) = \frac{1}{n}\sum_{j=1}^{n} \delta_{Z_j}, \text{ and, } \mathcal{Z}(p_n') = \frac{1}{n-1}\sum_{j=1}^{n-1} \delta_{y_j^{(n)}}.$$

Results for finite 1-energy measures

Theorem (Pemantle and Rivin, 2011)

If μ has finite 1-energy (that is, if μ satisfies the condition $\mathbb{E}\left(\frac{1}{|Z-W|}\right) < \infty$, where $Z, W \overset{i.i.d.}{\sim} \mu$), then, $\mathcal{Z}(p_n')$ converges weakly in probability to μ.

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- Pemantle and Rivin, in the same paper, conjectured that this theorem holds for all probability measures μ on the complex plane.
- μ fails to have finite 1-energy if concentrated on a set of dimension 1 (for example, on a circle).

Results for measures on the unit circle

Theorem (S., 2012)

Suppose μ is any probability measure on the unit circle, and $Z_1, Z_2, \dots$ are chosen I.I.D. from μ with $p_n(z) = (z - Z_1)(z - Z_2)\dots(z - Z_n)$. Then, the empirical distribution of the critical points of p_n, $\mathcal{Z}(p_n')$, converges weakly in probability to μ.

Results on distributions on the unit circle

Proposition (S., 2012)

Let $z_1, z_2, \ldots \in \mathbb{C}$ with $p_n(z) = (z - z_1)(z - z_2)\ldots(z - z_n)$. Let us denote the critical points of p_n by $y_1^{(n)}, y_2^{(n)}, \ldots, y_{n-1}^{(n)}$. Also, suppose z_j's satisfy the property

$$\frac{z_1^k + z_2^k + \ldots + z_n^k}{n} \longrightarrow c_k, \forall k \in \mathbb{N},$$

where $|c_k| < \infty, \forall k$. Then,

$$\frac{(y_1^{(n)})^k + (y_2^{(n)})^k + \ldots + (y_{n-1}^{(n)})^k}{n-1} \longrightarrow c_k, \forall k \in \mathbb{N}.$$

Results on distributions on the unit circle

Generalization of the theorem in [Pemantle and Rivin, 2011], about convergence of $\mathcal{Z}(p_n')$ in probability to the circle:

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If μ is any probability measure on the unit circle, then the empirical distribution of the critical points, $\mathcal{Z}(p_n')$, converges to the unit circle in probability.

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If μ is any probability measure on the unit circle, then the empirical distribution of the critical points, $\mathcal{Z}(p_n')$, converges to the unit circle in probability.
In fact, if μ is not uniform, this convergence is an almost sure convergence.

Any probability measure μ?

Theorem (Kabluchko, 2012)

Suppose μ is any probability measure on $\mathbb{C}$, and $Z_1, Z_2, ...$ are chosen I.I.D. from μ with $p_n(z) = (z - Z_1)(z - Z_2)...(z - Z_n)$. Then, the empirical distribution of the critical points of p_n, $\mathcal{Z}(p_n')$, converges weakly in probability to μ.

Zeros that take values ± 1 with equal probability

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Suppose $X_j, j \in \mathbb{N}$, are I.I.D. with $\mathbb{P}(X_j = +1) = \mathbb{P}(X_j = -1) = \frac{1}{2}$.

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Let $f_N(z) = \prod_{j=1}^{N} \left(1 - \frac{z}{X_j}\right)$ - a random polynomial vanishing at X_j's.

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$$e_{k,N} = \sum_{1 \le j_1 < j_2 < \ldots < j_k \le N} X_{j_1} X_{j_2} \ldots X_{j_k}.$$

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We wish to study the behavior of these elementary symmetric polynomials.

Behavior of a normalized $e_{k,N}$ for varying k and fixed N

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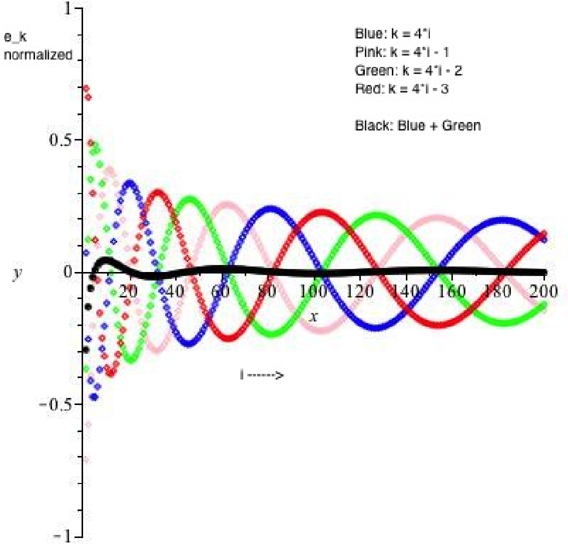

Explanation of sinusoidal behavior

Theorem (Pemantle and S., 2014)

Let $X_1, X_2, \ldots$ be i.i.d. Rademacher random variables. Let $e_{k,N}$ be the kth elementary symmetric function of $X_1, X_2, \ldots X_N$, $\alpha = \frac{X_1+X_2+\ldots+X_N}{\sqrt{N}}$ and $\Theta = \arctan\left(\frac{e_{k,N}}{\sqrt{\frac{k}{N}}e_{k+1,N}}\right)$. Then, if d_H is the Hausdorff distance between

$$\mathcal{Y} := \left\{\left(t, -\sin\left(\frac{t\alpha}{4} - \Theta\right)\right) : 0 \leq t \leq M\right\}$$

and

$$\Xi := \left\{\left(t, \frac{e_{k+t\sqrt{k},N}}{\sqrt{e_{k,N}^2 + \frac{k}{N}e_{k+1,N}^2}} \cdot \left(\frac{k}{N}\right)^{t\sqrt{k}/2}\right) : t = 0, \frac{4}{\sqrt{k}}, \frac{8}{\sqrt{k}}, \ldots, \lfloor M \rfloor_0\right\}$$

where $\lfloor M \rfloor_0$ is the highest value that is $\leq M$ and equals a multiple of $4/\sqrt{k}$, and M is any positive integer, then,

$$d_H \xrightarrow{P} 0$$

as $k \to \infty, N \to \infty$ under the constraint that $N/k^2 \to \infty$.

Main method

Cauchy's integral formula: For a simple loop Γ around 0,

$$f_N^{(k)}(0) = \frac{k!}{2\pi i}\int_\Gamma \frac{f_N(z)}{z^{k+1}}dz,$$

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The trick is to choose a Γ "of steepest descent" by finding where the saddle points of $f_N(z)/z^k$ are.

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Lemma

$\sigma_{k,N} \sim i\sqrt{\frac{k}{N}}$ in probability, and likewise, $\overline{\sigma_{k,N}} \sim -i\sqrt{\frac{k}{N}}$ in probability, as $N \to \infty, k \to \infty, N/k \to \infty$.

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[For the rest of this talk, we shall use σ and $\overline{\sigma}$ instead of $\sigma_{k,N}$ and $\overline{\sigma_{k,N}}$, for ease of notation.]

Main method

Proposition (Pemantle and S., 2014)

$$e_{k,N} = (-1)^k \Re\left\{(1+\eta_{k,N})\frac{f_N(\sigma)}{\sigma^k}\frac{1}{\sqrt{\pi k}}\right\}$$
$$= \left(\frac{N-k}{k}\right)^{k/2}\Re\left\{i^k(1+\eta_{k,N})f_N(\sigma)e^{-ik\theta}\frac{1}{\sqrt{\pi k}}\right\},$$

where $\eta_{k,N} \to 0$ *as* $N \to \infty, k \to N/k^2 \to \infty$ *where* $\theta = \arg(\sigma - \pi/2)$.

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This expression for the symmetric polynomials of the i.i.d. Rademacher random variables lead to the theorem regarding their sinusoidal behavior.

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We consider, for $x \in \mathbb{R}$,

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Does $f_x(z)$ exist? Is it analytic in some domain? Is it entire?

Poisson points of intensity 1 on $\mathbb{R}$

Lemma (Pemantle and S., *submitted*, 2014)

For all $x \in \mathbb{C}$, the product in $f_x(z)$ converges, with probability 1, uniformly in compact subsets of $\mathbb{C}$, making f_x an entire function.

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Write $f(z) := f_0(z) = \lim_{N\to\infty} \prod_{|X_j|\le N} \left(1 - \frac{z}{X_j}\right)$.

Elementary Symmetric Functions of $\frac{1}{X_j}$'s

The coefficient of z^k in f is $(-1)^k$ times the elementary symmetric functions

$$e_k := \lim_{N\to\infty} e_{k,N} = \lim_{N\to\infty} \sum_{j_1<j_2<\ldots<j_k : |X_{j_i}|\leq N} \frac{1}{X_{j_1}X_{j_2}\ldots X_{j_k}}.$$

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As before, we can use Cauchy's integral formula:

$$e_k = \frac{(-1)^k}{2\pi i}\int_\Gamma \frac{f(z)}{z^{k+1}}dz,$$

Γ being any simple loop around the origin.

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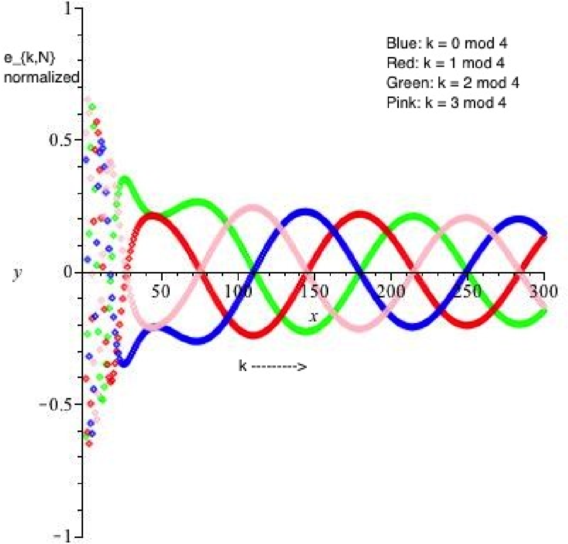

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Theorem (Pemantle and S., *submitted*, 2014)

Let $E_{M,k}$ be the event that the logarithmic derivative of $\frac{f(z)}{z^k}$ has a unique zero in a ball of radius $M\sqrt{k}$ about ik/π. Then $\mathbb{P}(E_{M,k}) \to 1$ as $M, k \to \infty$ with $k \geq 4\pi^2 M^2$.

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So we choose σ_k and $\overline{\sigma_k}$ as per the above theorem, and let Γ be the circle centered at 0 with radius $|\sigma_k|$.

Expression for e_k

Theorem (Pemantle and S., *submitted*, 2014)

For fixed r,

$$e_{k+r} = 2(-1)^{k+r}\Re\left\{(1+o(1))\,\frac{f_N(\sigma_k)}{\sigma_k^{k+r}}\sqrt{\frac{1}{2\pi k}}\right\},$$

in probability as $k \to \infty$.

Two step ratio for the e_k's

Proposition (Pemantle and S., *submitted*, 2014)

If $\{X_j\}_j$ denote the points of a Poisson process of intensity 1 on $\mathbb{R}$, and e_k is the kth elementary symmetric function of $1/X_j$'s, then

$$\frac{k^2 e_{k+2}}{e_k} \xrightarrow{P} -\pi^2.$$

Resultant zero set on repeatedly differentiating f

Theorem (Pemantle and S., *submitted*, 2014)

If $\{X_j\}_j$ *denote the points of a Poisson process of intensity 1 on* $\mathbb{R}$, *and*

$$f(z) = \lim_{N\to\infty} \prod_{|X_j|\leq N} \left(1 - \frac{z}{X_j}\right),$$

then, the zero set of the nth derivative, $f^{(n)}$, *of* f *converges as* $n \to \infty$ *in distribution to* $\mathbb{Z} + \mathcal{U}$, *where* $\mathcal{U} \sim$ *Uniform*$(0, 1)$.

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Thank You!