Zeros of the derivatives of random polynomials and random entire functions

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Overview

1. Introduction

2. Critical points of a complex polynomial with I.I.D zeros

3. Coefficients of a random polynomial with Rademacher zeroes

4. Random entire function vanishing at Poisson points
Random Polynomials of degree $n$

Create a random polynomial of degree $n$ by placing some probability measure on the space of all complex polynomials of degree $n$. Most obvious kind of random polynomial is the one where the coefficients are independent with some distribution:

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n,$$

where $c_j$'s are independent random variables. Another class of random polynomials arise as the characteristic polynomials of random matrices. Traditionally, we ask questions about the zeros of these random polynomials.
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- Another class of random polynomials arise as the characteristic polynomials of random matrices.
- Traditionally, we ask questions about the zeros of these random polynomials.
Mark Kac (1943) showed that, if the coefficients $c_j$'s are IID standard normal (real), then the expected number of real zeros are $\sim \frac{2}{\pi} \log n$. 

More recently, Kabluchko and Zaporozhets (2012) derived the asymptotic empirical distribution of the zeros of elliptic random polynomials and Littlewood-Offord random polynomials under a mild condition.
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Random Analytic Functions

Traditional ways to think of a random analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_j$'s are independent random variables.

Gaussian analytic functions on some domain $\Lambda$: A function $f$ such that $(f(z_1), \ldots, f(z_n))$ has a mean zero complex Gaussian distribution for every $n \geq 1$ and every $z_1, \ldots, z_n \in \Lambda$. 
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The number of zeros of $f(z) = \sum a_n z^n$, where $a_n$ are i.i.d. complex Gaussian, in a disk of radius $r$ about the origin has the same distribution as the sum of independent $\{0, 1\}$-valued random variables $X_k$, where $\mathbb{P}(X_k = 1) = r^{2k}$ - Peres and Virág (2005).
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Kabluchko and Zaporozhets (2012) derived the asymptotic empirical distribution of the zeros of flat analytic functions and hyperbolic analytic functions.
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Zeros and Critical Points and Random Functions

QUESTION - A new class of random polynomials and random entire functions

Suppose, instead of the traditional approach of having a probability measure on the coefficients of the polynomial / power series, we now consider probability measures on the zeros. Can we say anything about the location or distribution of critical points of these functions? What about their coefficients?
Our Random Polynomials

A random polynomial of degree $n$ is given by:

$$p_n(z) = (z - Z_1)(z - Z_2) \ldots (z - Z_n),$$

where $Z_j$'s are independent random variables.
Our Random Polynomials

Random polynomial of degree $n$

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Our Random Analytic Functions

If \( \{Z_n\} \) is a sequence of random variables (assume that, with probability 1 there are no cluster points), then we can use Weierstrass Product Formula to construct an entire function which vanishes at exactly these points. But there is a small problem: there are infinitely many entire functions that have the same zeros!
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But there is a small problem: there are infinitely many entire functions that have the same zeros!
Suppose \( Z_1, Z_2, \ldots \) are chosen I.I.D. using some probability measure \( \mu \) from the complex plane. Define \( p_n(z) = (z - Z_1)(z - Z_2) \cdots (z - Z_n) \).

Let \( y(n) \) be the critical points of \( p_n \) and \( Z(\lfloor f \rfloor) \) be the empirical distribution of the roots of a function \( f \).

Thus, \( Z(p_n) = \frac{1}{n} \sum_{j=1}^{n} \delta_{Z_j} \), and, \( Z(p'_n) = \frac{1}{n-1} \sum_{j=1}^{n-1} \delta_{y(n)_j} \).
Suppose $Z_1, Z_2, ...$ are chosen I.I.D. using some probability measure $\mu$ from the complex plane. Define

$$p_n(z) = (z - Z_1)(z - Z_2)...(z - Z_n).$$
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$$p_n(z) = (z - Z_1)(z - Z_2)...(z - Z_n).$$

Let $y_1^{(n)}, y_2^{(n)}, ..., y_{n-1}^{(n)}$ be the critical points of $p_n$ and $\mathcal{Z}(f)$ be the empirical distribution of the roots of a function $f$. 
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$$\mathcal{Z}(p_n) = \frac{1}{n} \sum_{j=1}^{n} \delta_{Z_j}, \text{ and, } \mathcal{Z}(p'_n) = \frac{1}{n-1} \sum_{j=1}^{n-1} \delta_{y_j^{(n)}}.$$
Results for finite 1-energy measures

Theorem (Pemantle and Rivin, 2011)

If \( \mu \) has finite 1-energy (that is, if \( \mu \) satisfies the condition
\[
E \left( \frac{1}{|Z-W|} \right) < \infty, \quad \text{where } Z, W \overset{i.i.d.}{\sim} \mu, \quad \text{then, } Z(p_n') \text{ converges weakly in probability to } \mu.
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- Pemantle and Rivin, in the same paper, conjectured that this theorem holds for all probability measures $\mu$ on the complex plane.
Critical points of a complex polynomial with I.I.D zeros

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- Pemantle and Rivin, in the same paper, conjectured that this theorem holds for all probability measures $\mu$ on the complex plane.
- $\mu$ fails to have finite 1-energy if concentrated on a set of dimension 1 (for example, on a circle).
Theorem (S., 2012)

Suppose $\mu$ is any probability measure on the unit circle, and $Z_1, Z_2, \ldots$ are chosen I.I.D. from $\mu$ with $p_n(z) = (z - Z_1)(z - Z_2)\ldots(z - Z_n)$. Then, the empirical distribution of the critical points of $p_n$, $\mathcal{Z}(p_n')$, converges weakly in probability to $\mu$. 
Proposition (S., 2012)

Let \( z_1, z_2, \ldots \in \mathbb{C} \) with \( p_n(z) = (z - z_1)(z - z_2) \ldots (z - z_n) \). Let us denote the critical points of \( p_n \) by \( y_1^{(n)}, y_2^{(n)}, \ldots, y_{n-1}^{(n)} \). Also, suppose \( z_j \)'s satisfy the property

\[
\frac{z_1^k + z_2^k + \ldots + z_n^k}{n} \to c_k, \quad \forall k \in \mathbb{N},
\]

where \( |c_k| < \infty, \forall k \). Then,

\[
\frac{(y_1^{(n)})^k + (y_2^{(n)})^k + \ldots + (y_{n-1}^{(n)})^k}{n-1} \to c_k, \quad \forall k \in \mathbb{N}.
\]
Generalization of the theorem in [Pemantle and Rivin, 2011], about convergence of $\mathcal{Z}(p'_n)$ in probability to the circle:

Lemma (S., 2012)

*If $\mu$ is any probability measure on the unit circle, then the empirical distribution of the critical points, $\mathcal{Z}(p'_n)$, converges to the unit circle in probability.*
Results on distributions on the unit circle

Generalization of the theorem in [Pemantle and Rivin, 2011], about convergence of $\mathcal{Z}(p'_n)$ in probability to the circle:

Lemma (S., 2012)

If $\mu$ is any probability measure on the unit circle, then the empirical distribution of the critical points, $\mathcal{Z}(p'_n)$, converges to the unit circle in probability.

In fact, if $\mu$ is not uniform, this convergence is an almost sure convergence.
Any probability measure $\mu$?

**Theorem (Kabluchko, 2012)**

Suppose $\mu$ is any probability measure on $\mathbb{C}$, and $Z_1, Z_2, ...$ are chosen i.i.d. from $\mu$ with $p_n(z) = (z - Z_1)(z - Z_2)...(z - Z_n)$. Then, the empirical distribution of the critical points of $p_n$, $\mathcal{Z}(p_n')$, converges weakly in probability to $\mu$. 
Zeros that take values $\pm 1$ with equal probability
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Suppose $X_j, j \in \mathbb{N}$, are i.i.d. with $\mathbb{P}(X_j = +1) = \mathbb{P}(X_j = -1) = \frac{1}{2}$. 
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Let $f_N(z) = \prod_{j=1}^{N} \left( 1 - \frac{z}{X_j} \right)$ - a random polynomial vanishing at $X_j$'s.
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$$e_{k,N} = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq N} X_{j_1} X_{j_2} \ldots X_{j_k}.$$
Zeros that take values $\pm 1$ with equal probability

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We wish to study the behavior of these elementary symmetric polynomials.
Behavior of a normalized $e_{k,N}$ for varying $k$ and fixed $N$
Behavior of a normalized $e_{k,N}$ for varying $k$ and fixed $N$
Explanation of sinusoidal behavior

**Theorem (Pemantle and S., 2014)**

Let $X_1, X_2, ...$ be i.i.d. Rademacher random variables. Let $e_{k,N}$ be the $k$th elementary symmetric function of $X_1, X_2, ... X_N$, $\alpha = \frac{X_1 + X_2 + ... + X_N}{\sqrt{N}}$ and $\Theta = \arctan\left(\frac{e_{k,N}}{\sqrt{k} e_{k+1,N}}\right)$. Then, if $d_H$ is the Hausdorff distance between

$$\mathcal{Y} := \left\{ \left( t, - \sin\left( \frac{t\alpha}{4} - \Theta \right) \right) : 0 \leq t \leq M \right\}$$

and

$$\Xi := \left\{ \left( t, \frac{e_{k+\sqrt{k},N}}{\sqrt{e_{k,N}^2 + \frac{k}{N} e_{k+1,N}^2}} \cdot \left( \frac{k}{N} \right)^{t\sqrt{k}/2} \right) : t = 0, \frac{4}{\sqrt{k}}, \frac{8}{\sqrt{k}}, ..., \lfloor M \rfloor \right\}$$

where $\lfloor M \rfloor$ is the highest value that is $\leq M$ and equals a multiple of $4/\sqrt{k}$, and $M$ is any positive integer, then,

$$d_H \xrightarrow{P} 0$$

as $k \to \infty$, $N \to \infty$ under the constraint that $N/k^2 \to \infty$. 

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Critical points of random analytic functions
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Cauchy’s integral formula: For a simple loop $\Gamma$ around 0,

$$f_N^{(k)}(0) = \frac{k!}{2\pi i} \int_{\Gamma} \frac{f_N(z)}{z^{k+1}} dz,$$
Main method

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$$\implies k!(-1)^k e_{k,N} = \frac{k!}{2\pi i} \int_{\Gamma} \frac{f_N(z)}{z^{k+1}} dz,$$
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The trick is to choose a $\Gamma$ “of steepest descent” by finding where the saddle points of $f_N(z)/z^k$ are.
Main method

Suppose $\sigma_{k,N}$ and $\overline{\sigma_{k,N}}$ are the critical points of $\log \left( \frac{f_N(z)}{z^k} \right)$.
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Suppose $\sigma_{k,N}$ and $\overline{\sigma_{k,N}}$ are the critical points of $\log \left( \frac{f_N(z)}{z^k} \right)$. We choose $\Gamma$ to be the circle centered at 0 with radius $|\sigma_{k,N}|$. 

[For the rest of this talk, we shall use $\sigma$ and $\overline{\sigma}$ instead of $\sigma_{k,N}$ and $\overline{\sigma_{k,N}}$, for ease of notation.]
Suppose $\sigma_{k,N}$ and $\overline{\sigma_{k,N}}$ are the critical points of $\log \left( \frac{f_N(z)}{z^k} \right)$.
We choose $\Gamma$ to be the circle centered at 0 with radius $|\sigma_{k,N}|$.

**Lemma**

$$\sigma_{k,N} \sim i \sqrt{\frac{k}{N}} \text{ in probability, and likewise, } \overline{\sigma_{k,N}} \sim -i \sqrt{\frac{k}{N}} \text{ in probability, as } N \to \infty, k \to \infty, N/k \to \infty.$$
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Main method

Proposition (Pemantle and S., 2014)

\[
e_{k,N} = (-1)^k \Re \left\{ (1 + \eta_{k,N}) \frac{f_N(\sigma)}{\sigma^k} \frac{1}{\sqrt{\pi k}} \right\}
= \left( \frac{N - k}{k} \right)^{k/2} \Re \left\{ i^k (1 + \eta_{k,N}) f_N(\sigma) e^{-ik\theta} \frac{1}{\sqrt{\pi k}} \right\},
\]

where \( \eta_{k,N} \to 0 \) as \( N \to \infty \), \( k \to N/k^2 \to \infty \) where \( \theta = \arg(\sigma - \pi/2) \).
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where \( \eta_{k,N} \rightarrow 0 \) as \( N \rightarrow \infty, k \rightarrow N/k^2 \rightarrow \infty \) where \( \theta = \arg(\sigma - \pi/2) \).

This expression for the symmetric polynomials of the i.i.d. Rademacher random variables lead to the theorem regarding their sinusoidal behavior.
Random entire function vanishing at Poisson points

Poisson points of intensity 1 on $\mathbb{R}$

Let $X_j, j \in \mathbb{N}$, denote the points of a Poisson process of intensity 1 on the real line.

We consider, for $x \in \mathbb{R}$,

$$f_x(z) := \lim_{N \to \infty} \prod_{|X_j - x| \leq N} (1 - z X_j).$$

Does $f_x(z)$ exist? Is it analytic in some domain? Is it entire?
Let $X_j, j \in \mathbb{N}$, denote the points of a Poisson process of intensity 1 on the real line.
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Does $f_x(z)$ exist? Is it analytic in some domain? Is it entire?
Lemma (Pemantle and S., submitted, 2014)

For all \( x \in \mathbb{C} \), the product in \( f_x(z) \) converges, with probability 1, uniformly in compact subsets of \( \mathbb{C} \), making \( f_x \) an entire function.
Lemma (Pemantle and S., submitted, 2014)

For all $x \in \mathbb{C}$, the product in $f_x(z)$ converges, with probability 1, uniformly in compact subsets of $\mathbb{C}$, making $f_x$ an entire function. Moreover, for each $k$, the law of the zero set of $f_x^{(k)}(z)$ does not depend on $x$. 

$\text{Write } f(z) := f_0(z) = \lim_{N \to \infty} \prod_{|X_j| \leq N} (1 - zX_j).$

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Write $f(z) := f_0(z) = \lim_{N \to \infty} \prod_{|X_j| \leq N} \left( 1 - \frac{z}{X_j} \right)$. 
The coefficient of $z^k$ in $f$ is $(-1)^k$ times the elementary symmetric functions

$$ e_k := \lim_{N \to \infty} e_{k,N} = \lim_{N \to \infty} \sum_{j_1 < j_2 < \ldots < j_k : |X_{j_i}| \leq N} \frac{1}{X_{j_1} X_{j_2} \ldots X_{j_k}}. $$
Elementary Symmetric Functions of $\frac{1}{X_j}$'s

The coefficient of $z^k$ in $f$ is $(-1)^k$ times the elementary symmetric functions

$$e_k := \lim_{N \to \infty} e_{k,N} = \lim_{N \to \infty} \sum_{j_1 < j_2 < \ldots < j_k \mid |X_{j_i}| \leq N} \frac{1}{X_{j_1} X_{j_2} \ldots X_{j_k}}.$$

As before, we can use Cauchy’s integral formula:

$$e_k = \frac{(-1)^k}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{k+1}} dz,$$

$\Gamma$ being any simple loop around the origin.
Behavior of a normalized $e_{k,N}$ for varying $k$ and fixed $N$
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Main method

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Let $E_{M,k}$ be the event that the logarithmic derivative of $\frac{f(z)}{z^k}$ has a unique zero in a ball of radius $M\sqrt{k}$ about $ik/\pi$. Then $\mathbb{P}(E_{M,k}) \to 1$ as $M, k \to \infty$ with $k \geq 4\pi^2 M^2$. 
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Again, we let $\sigma_k$ and $\overline{\sigma_k}$ be critical points of $\log \left( \frac{f(z)}{z^k} \right)$. But wait, there are too many of them!

**Theorem (Pemantle and S., submitted, 2014)**

Let $E_{M,k}$ be the event that the logarithmic derivative of $\frac{f(z)}{z^k}$ has a unique zero in a ball of radius $M \sqrt{k}$ about $ik/\pi$. Then $\mathbb{P}(E_{M,k}) \to 1$ as $M, k \to \infty$ with $k \geq 4\pi^2 M^2$.

So we choose $\sigma_k$ and $\overline{\sigma_k}$ as per the above theorem, and let $\Gamma$ be the circle centered at 0 with radius $|\sigma_k|$.
Expression for $e_k$

Theorem (Pemantle and S., submitted, 2014)

For fixed $r$,

$$e_{k+r} = 2(-1)^{k+r} \Re \left\{ (1 + o(1)) \frac{f_N(\sigma_k)}{\sigma_k^{k+r}} \sqrt{\frac{1}{2\pi k}} \right\},$$

in probability as $k \to \infty$. 
Proposition (Pemantle and S., *submitted*, 2014)

If \( \{X_j\}_j \) denote the points of a Poisson process of intensity 1 on \( \mathbb{R} \), and \( e_k \) is the \( k \)th elementary symmetric function of \( 1/X_j \)'s, then

\[
\frac{k^2 e_{k+2}}{e_k} \xrightarrow{P} -\pi^2.
\]
Theorem (Pemantle and S., submitted, 2014)

If \( \{X_j\}_j \) denote the points of a Poisson process of intensity 1 on \( \mathbb{R} \), and

\[
    f(z) = \lim_{N \to \infty} \prod_{|X_j| \leq N} \left(1 - \frac{z}{X_j}\right),
\]

then, the zero set of the \( n \)th derivative, \( f^{(n)} \), of \( f \) converges as \( n \to \infty \) in distribution to \( \mathbb{Z} + \mathcal{U} \), where \( \mathcal{U} \sim \text{Uniform}(0, 1) \).
References

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  *arXiv:1409.7956*

- **Sneha Subramanian (2012)**
  On the distribution of critical points of a polynomial
  
Thank You!